V V COLLEGE OF ENGINEERING

(Approved By AICTE, New Delhi and Affiliated To Anna University Chennai)

V V Nagar, Arasoor ,Tisaiyanvilai

DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

EC8553 – DISCRETE TIME SIGNAL PROCESSING

(As per R2017 Regulation of Anna University, Chennai)

Year/ Semester: III / V

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EC8553 DISCRETE-TIME SIGNAL PROCESSING L T P C 4 0 0 4

OBJECTIVES:

To learn discrete fourier transform, properties of DFT and its application to linear filtering

To understand the characteristics of digital filters, design digital IIR and FIR filters and apply these filters to filter undesirable signals in various frequency bands

To understand the effects of finite precision representation on digital filters

To understand the fundamental concepts of multi rate signal processing and its applications

To introduce the concepts of adaptive filters and its application to communication engineering

UNIT I DISCRETE FOURIER TRANSFORM 12

Review of signals and systems, concept of frequency in discrete-time signals, summary of analysis & synthesis equations for FT & DTFT, frequency domain sampling, Discrete Fourier transform (DFT) - deriving DFT from DTFT, properties of DFT - periodicity, symmetry, circular convolution. Linear filtering using DFT. Filtering long data sequences - overlap save and overlap add method. Fast computation of DFT - Radix-2 Decimation-in-time (DIT) Fast Fourier transform (FFT), Decimation-in-frequency (DIF) Fast Fourier transform (FFT). Linear filtering using FFT.

UNIT II INFINITE IMPULSE RESPONSE FILTERS 12

Characteristics of practical frequency selective filters. characteristics of commonly used analog filters - Butterworth filters, Chebyshev filters. Design of IIR filters from analog filters (LPF, HPF, BPF, BRF) - Approximation of derivatives, Impulse invariance method, Bilinear transformation. Frequency transformation in the analog domain. Structure of IIR filter - direct form I, direct form II, Cascade, parallel realizations.

UNIT III FINITE IMPULSE RESPONSE FILTERS 12

Design of FIR filters - symmetric and Anti-symmetric FIR filters - design of linear phase FIR filters using Fourier series method - FIR filter design using windows (Rectangular, Hamming and Hanning window), Frequency sampling method. FIR filter structures - linear phase structure, direct form realizations

UNIT IV FINITE WORD LENGTH EFFECTS 12

Fixed point and floating point number representation - ADC - quantization - truncation and rounding - quantization noise - input / output quantization - coefficient quantization error – product quantization error - overflow error - limit cycle oscillations due to product quantization and summation - scaling to prevent overflow.

UNIT V INTRODUCTION TO DIGITAL SIGNAL PROCESSORS 12

DSP functionalities - circular buffering – DSP architecture – Fixed and Floating point architecture principles – Programming – Application examples.

TOTAL: 60 PERIODS

OUTCOMES:

At the end of the course, the student should be able to

Apply DFT for the analysis of digital signals and systems

Design IIR and FIR filters

Characterize the effects of finite precision representation on digital filters

Design multirate filters

Apply adaptive filters appropriately in communication systems

TEXT BOOK:

1. John G. Proakis & Dimitris G.Manolakis, “Digital Signal Processing – Principles, Algorithms & Applications”, Fourth Edition, Pearson Education / Prentice Hall, 2007. (UNIT I – V)

REFERENCES:

1. Emmanuel C. Ifeachor & Barrie. W. Jervis, “Digital Signal Processing”, Second Edition, Pearson Education / Prentice Hall, 2002.

2. A. V. Oppenheim, R.W. Schafer and J.R. Buck, “Discrete-Time Signal Processing”, 8^{th} Indian Reprint, Pearson, 2004.

3. Sanjit K. Mitra, “Digital Signal Processing – A Computer Based Approach”, Tata Mc Graw Hill, 2007.

4. Andreas Antoniou, “Digital Signal Processing”, Tata Mc Graw Hill, 2006.

EC8553 / DISCRETE-TIME SIGNAL PROCESSING.

UNIT I. DISCRETE FOURIER TRANSFORM.

Review of Signals and Systems:

Signal - Any physical quantity that varies with time, space and other independent valiable.
- Function of one or more independent variables.

ex: Speech signal, ECG, EEG etc.

Types: 1) i) Continuous-time signals:- These signals are defined for every instant of time. (ie) $x(t)$.
ii) Discrete-time Signals:- These signals are defined at discrete instants of time. (ie) $x(n)$
- Continuous in amplitude, discrete in time

Example: Sketch the continuous signal $x(t)=2e^{-2t}$ for an interval $0 \le t \le 2$. Sample the continuous signal with the sampling period $T=0.2$ sec and sketch discrete time signal.

$$x(t) = 2e^{-2t}.$$

$x(0)=2$, $x(0.2)=1.3406$, $x(0.4)=0.8987$
$x(0.6)=0.6024$, $x(0.8)=0.4038$, $x(1)=0.2707$

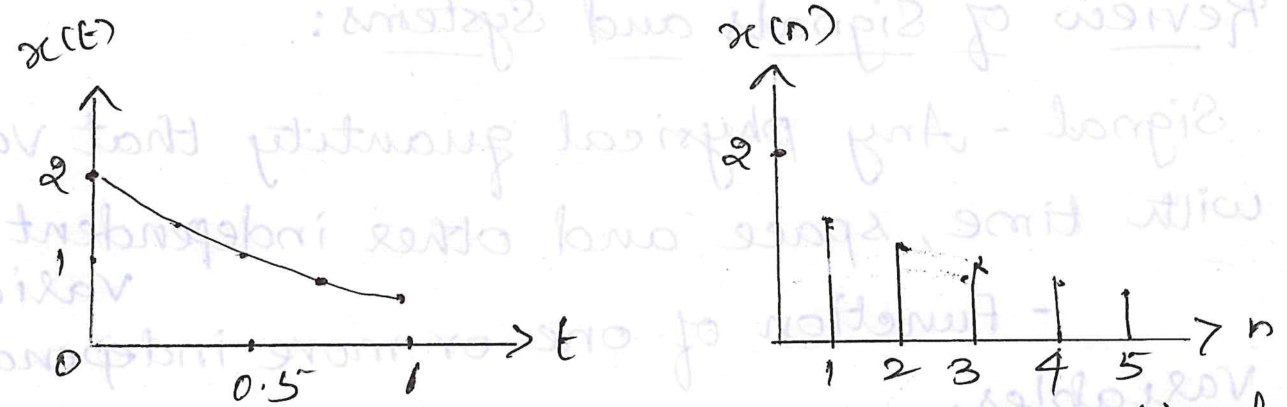
Given sampling period $T=0.2$ sec.

$$x(nT) = x(t)\Big|_{t=nT} \quad [t=0.2n]$$

$x(0.2n) = 2e^{-2(0.2n)} = 2e^{-0.4n}$

$x(0) = 2$, $x(1) = 1.3406$, $x(2) = 0.8987$

$x(3) = 0.6024$, $x(4) = 0.4038$, $x(5) = 0.2707$



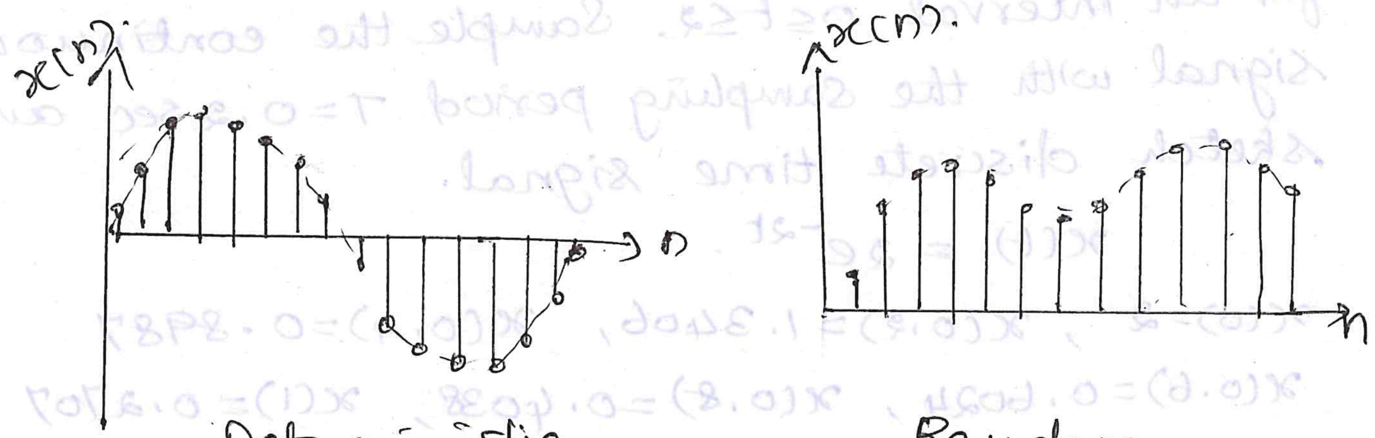
Continuous time Signal. Discrete time signal.

2) i) Deterministic Signal:

- Signal exhibiting no uncertainty of value at any given instant of time. ex. $x(n) = \sin \pi n$.

ii) Random Signal:

- Signal characterized by uncertainty before its actual occurance. ex. noise.

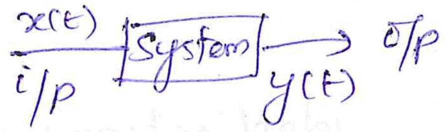


Deterministic Random.

System:

- Interconnection of components.
- physical device that performs an operation on an input signal and produces another signal as output.

(ie) It is a physical device that generates a response or o/p signal for a given input signal.



ex: Communication system

Manufacturing system.

Amplifier - an electronic system.

Continuous time system:

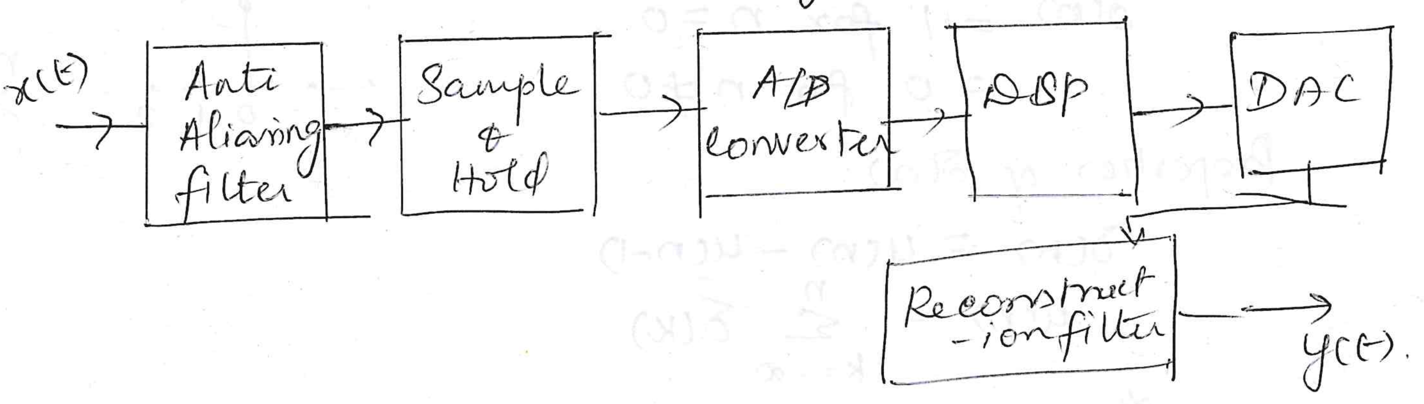
- operates on a continuous time signal and produces a continuous time output signal.

$$y(t) = T[x(t)]$$

Discrete-time system:

- Operates on a discrete-time signal and produces discrete time output signal.

Digital Signal Processing System:



Advantages: 1) Greater Accuracy
2) Cheaper
3) Ease of Storage
4) Flexibility
5) Time Sharing.

Limitations: 1) System complexity 2) Power Consumption
3) Bandwidth limited by sampling rate.

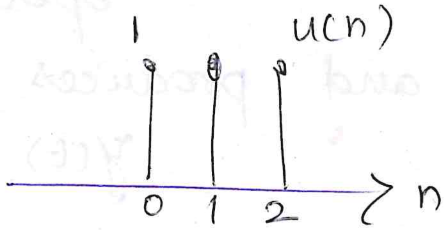
Applications: 1) Telecommunication, 2) Medicine
3) Speech processing 4) Military
5) Instrumentation & control etc.

Elementary Discrete-time Signals:

1) Unit Step Sequence:

$u(n) = 1$ for $n \geq 0$

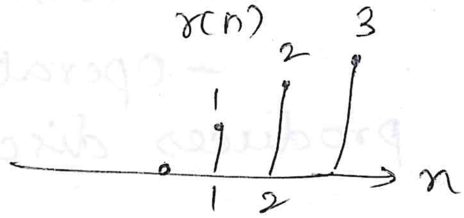
$\quad\;\; 0$ for $n < 0$



2) Unit ramp sequence:

$r(n) = n$ for $n \geq 0$

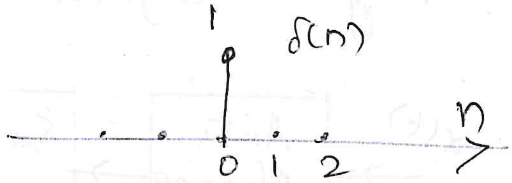
$\quad\;\; = 0$ for $n < 0$



3) Unit impulse response:

$\delta(n) = 1$ for $n = 0$

$\quad\;\; = 0$ for $n \neq 0$



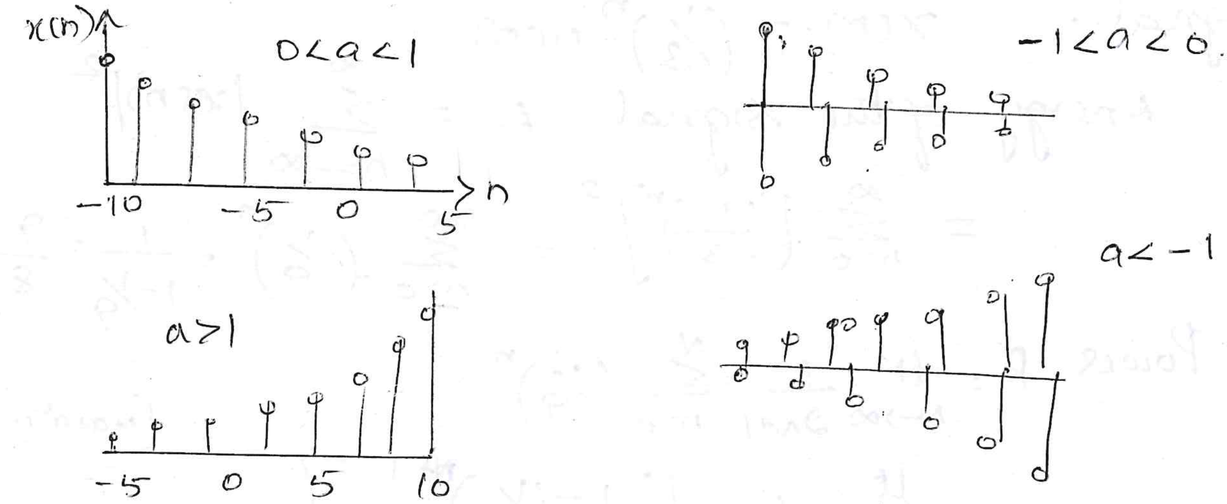
Properties of $\delta(n)$:

$$\delta(n) = u(n) - u(n-1)$$

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

$$\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$$

4) Exponential Sequence: $x(n) = a^n$ for all n.



Problems: Find the following summations.

1) $\sum_{n=-\infty}^{\infty} \delta(n-2) \sin 2n$ $\left[\delta(n-2) = 1 \text{ for } n=2 \atop = 0 \text{ otherwise}\right.$

$= \sin 2n \Big|_{n=2} = \sin 4.$

2) $\sum_{n=0}^{\infty} \delta(n)\, e^{2n} = e^{2n}\Big|_{n=0} = 1$

3) $\sum_{n=-\infty}^{\infty} \delta(n+1)\, x(n) = x(n)\Big|_{n=-1} = x(-1).$

4) $\sum_{n=0}^{\infty} \delta(n+1)\, e^{-2n} = 0$ [because for $n=-1$ $\delta(n+1) = 1$ but it is not within limit

classification of discrete time signals:

1) Energy and Power Signals:

Energy $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

Power $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2.$

* A signal is an energy sig., if total energy is finite. For an energy signal, $P = 0$.
* A sig. is power sig, if ~~total~~ Average power is finite. For power signal, $E = \infty$.

Problem: Find the values of power & energy of the signal. $x(n) = \left(\frac{1}{3}\right)^n u(n)$

Energy of the signal $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{3}\right)^n\right]^2 = \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

Power $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$ $\quad \left[1+a+a^2+\dots\infty = \frac{1}{1-a}\right]$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{9}\right)^{N+1}}{1-\frac{1}{9}}\right] = 0$$

The energy is finite & power is zero. $\therefore$ Energy Signal

2) Periodic and aperiodic signals:

Sig. Periodic with period N means

$x(N+n) = x(n)$ for all n.

ex: 1) $x(n) = e^{j6\pi n}$.

$\omega_0 = 6\pi$. Fundamental frequency is multiple of π. $\therefore$ signal is periodic.

$$N = 2\pi\left[\frac{m}{\omega_0}\right] = 2\pi\left[\frac{m}{6\pi}\right]$$

Min. value of m for which N is 3

$\therefore N = 2\pi\left[\frac{3}{6\pi}\right] = 1$, $\therefore$ Fundamental period $N = 1$.

2) $x(n) = e^{j\frac{3}{5}(n+\frac{1}{2})}$, $\omega_0 = 3/5$ which is not multiple of π. $\therefore$ Signal is aperiodic.

3) Symmetric (even) & Anti symmetric (odd) signals:

Even signal : $x(-n) = x(n)$ for all n.

Odd signal : $x(-n) = -x(n)$ for all n.

ex: $x(n) = \cos \omega n$ (even).

$x(n) = A \sin \omega n$ (odd).

If $x(n)$ is sum of odd & even components

$$x(n) = x_e(n) + x_o(n).$$

$$x(-n) = x_e(-n) + x_o(-n)$$

$$= x_e(n) - x_o(n).$$

$$\Rightarrow \quad 2x_e(n) = x(n) + x(-n)$$

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

$$\therefore x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

4) Causal and Non causal signals:

A signal $x(n)$ is said to causal if its value is zero for $n<0$. Otherwise non causal.

Causal : $x_1(n) = a^n u(n)$.

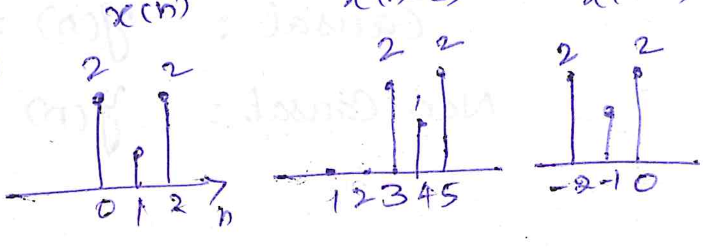
$$x_2(n) = \{\underset{\uparrow}{1}, 2, -3, -1\}$$

Non causal : $x_1(n) = a^n u(-n+1)$

$$x_2(n) = \{1, -2, \underset{\uparrow}{1}, 4, 3\}$$

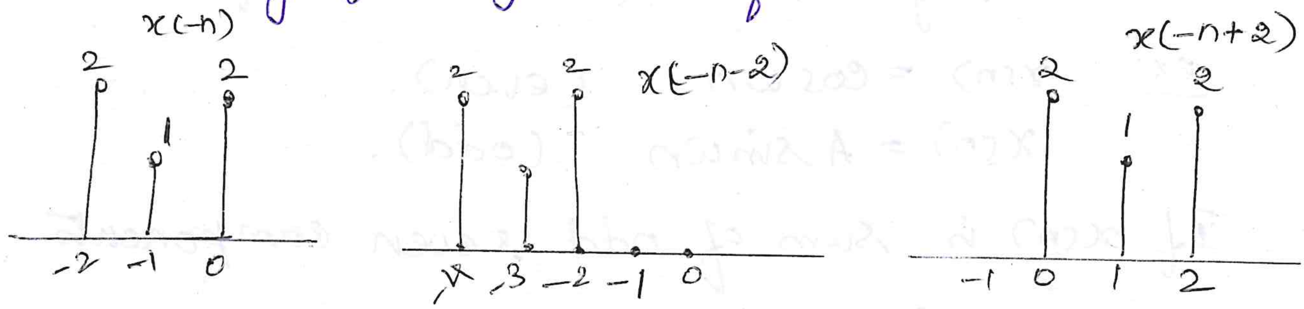
Operation on Signals:

1) Shifting : $y(n) = x(n-k)$



2) Time Reversal:

Time reversal of sequence $x(n)$ is obtained by folding the seq. about $n=0$, (ie) $x(-n)$



3) Time Scaling: Replace n by λn. $y(n) = x(\lambda n)$

4) Scalar multiplication: $y(n) = a\,x(n)$.

Classification of Discrete-time systems:

1) Static and Dynamic System:

Static (memoryless) - If its o/p at any instant depends on input samples but not on past or future samples of the input.

Static : $y(n) = a\,x(n)$, $y(n) = a\,x^2(n)$.

Dynamic : $y(n) = x(n-1) + x(n-2) + x(n+1)$

2) causal and non-causal systems:

Causal - o/p of system depends on present and past inputs but doesn't depend on future i/p.

Non causal - If o/p depends on future i/p, then anti causal.

Causal : $y(n) = x(n) + x(n-1)$

Non Causal : $y(n) = x(2n)$

3) Linear and non-linear systems:

Linear - System satisfies the superposition principle.
Superposition states that response of the system to a weighted sum of signals should be equal to corresponding weighted sum of the o/ps of the system to each of the individual i/p sig.

Linear: $T[a_1x_1(n) + a_2x_2(n)] = a_1T[x_1(n)] + a_2T[x_2(n)]$

4) Time variant and Time-invariant systems:

Time-invariant → characteristics of the system don't change with time.

If $y(n)$ is response to i/p $x(n)$, then response of the system to the i/p $x(n-k)$ is $y(n-k)$.

$$y(n,k) = T[x(n-k)]$$

Time invariant : $y(n,k) = y(n-k)$
Time variant : $y(n,k) \neq y(n-k)$

5) Stable and Unstable Systems:

LTI system is stable if it produces a bounded o/p for every bounded i/p.

Necessary and sufficient condition for stability is $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

If, for some bounded i/p sequence, the o/p is unbounded (infinite), then the system is unstable.

Introduction to DFT:

The Discrete Fourier Transform (DFT) is a powerful computation tool to evaluate the Fourier transform $X(e^{j\omega})$.

- DFT is defined only for sequences of finite length
- DFT of a sequence is periodic, in the range 0 to 2π.

To compute N equally spaced points over the interval $0 \le \omega \le 2\pi$, then N points should be located at

$$\omega_k = \frac{2\pi}{N} k, \quad k = 0, 1 \cdots N-1$$

These N equally spaced frequency samples of DTFT are known as DFT denoted by $X(K)$ is $W_N = e^{-j2}$

$$X(K) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}, \quad 0 \le K \le N-1.$$

DFT: $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \ 0 \le K \le N-1$; IDFT $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j2\pi kn/N}, \ 0 \le n \le N-$

Properties of DFT:

1) Periodicity: If $X(K)$ is N-point DFT of a finite duration sequence $x(n)$ then

$$x(n+N) = x(n) \text{ for all } n.$$
$$X(K+N) = X(K) \text{ for all } k.$$

2) Linearity: If two finite duration seq. $x_1(n)$ & $x_2(n)$ are linearly combined as $x_3(n) = a x_1(n) + b x_2(n)$.

then DFT of $x_3(n)$ is $X_3(k) = a X_1(K) + b X_2(K)$.

If $x_1(n)$ has length N_1 & $x_2(n)$ has length N_2 then max. length of $x_3(n)$ will be $N_3 = Max(N_1, N_2)$.

ex: If $N_2 < N_1$, $X_2(k)$ is DFT of seg. $x_2(n)$ augmented by $N_1 - N_2$ zeros.

Concept of frequency in discrete-time signals:

Discrete-time sine signal is

$x(n) = A\cos(\omega n + \theta) \quad -\infty < n < \infty \quad$ n - integer
- Sample No.

$A \rightarrow$ Amplitude, $\omega \rightarrow$ frequency, $\theta \rightarrow$ phase

$x(n) = A\cos(2\pi f n + \theta)$ where $\omega = 2\pi f$.

Properties: 1) A discrete-time sinusoid is periodic only if its frequency f is a rational number.

(ie) $x(n+N) = x(n)$ for all $n \rightarrow$ periodic.

$\cos[2\pi f_0(N+n)+\theta] = \cos(2\pi f_0 n + \theta)$

So, $2\pi f_0 N = 2K\pi \Rightarrow f_0 = \frac{K}{N} \rightarrow$ rational

2) Discrete-time sinusoids whose frequencies are seperated by an integer multiple of 2π are identical.

$\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta)$

$= \cos(\omega_0 n + \theta)$

$x_K(n) = A\cos(\omega_K n + \theta)$ where $\omega_K = \omega_0 + 2k\pi$

3) The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pi$ (or) $f = \frac{1}{2}$

$x(n) = A\cos(\omega n + \theta) = \frac{A}{2}e^{j(\omega n + \theta)} + \frac{A}{2}e^{-j(\omega n + \theta)}$

Freq. range 2π. $\therefore$ $0 \le \omega \le 2\pi$ or $-\pi \le \omega \le \pi$

Fundamental range : $(0 \le f \le 1$ or $-\frac{1}{2} \le f \le \frac{1}{2})$.

Summary of analysis & Synthesis equations of FT & DTFT:

Fourier Transform: (Continuous Sig)

Analysis equation: $X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$

Synthesis equation: $x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$

For Discrete-time signals, (DTFT)

Analysis eqn: $C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$

Synthesis eqn: $x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N}$.

For FT, Analysis eqn: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Synthesis eqn: $x(n) = \frac{1}{2\pi} \int_{-2\pi} X(\omega) e^{j\omega n} d\omega$.

DTFT is the member of FT family that operates on aperiodic, discrete signals. For N samples, as $N \to \infty$, time domain becomes aperiodic and freq. domain becomes continuous signal. This is DTFT, the FT that relates an aperiodic, discrete signal with periodic, continuous frequency spectrum

Frequency domain Sampling:

Sampling is performed by applying continuous time signal to ADC whose o/p is digital values.

Discrete Sig. $x(nT) = x(n) \quad -\infty < n < \infty$.

Sampling period $\to$ Time interval bet. successive samples

Sampling rate $\to \frac{1}{T} = F_s$.

Consider a signal $x(t) = \sin \Omega t$

$x(nT) = \sin \Omega nT = \sin \omega n$ $[\omega = \Omega T]$

For continuous time sig, freq range $-\infty < \Omega < \infty$.
For discrete time sig, freq. range $-\pi < \omega < \pi$.

O/p of ADC, $x(n) = \sin\left[\left(\Omega + \frac{2\pi M}{T}\right) nT\right]$, $M = 1, 2 \ldots$

$= \sin\left[(\Omega nT + 2\pi M n)\right] = \sin \Omega nT.$

Sequence $x(n)$ is obtained by sampling sine sig. of Ω rad/sec. represents sine wave at other frequencies $\left(\Omega + \frac{2\pi k}{T}\right)$. When sig. is sampled at a rate f_s sample/sec, we can't distinguish bet. samples of sine wave freqs f Hz & $(f + k f_s)$ Hz. Thus, an infinite no. of continuous time signals are represented by same set of samples.

P.NO 468 **Symmetry property of DFT:**

N-point sequence $x(n)$ & its DFT are both complex valued.

$x(n) = x_R(n) + j x_I(n) \quad 0 \le n \le N-1$

$X(k) = X_R(k) + j X_I(k) \quad 0 \le k \le N-1.$

$\therefore X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N}\right]$

$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N}\right]$

IDFT $\quad x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos \frac{2\pi kn}{N} - X_I(k) \sin \frac{2\pi kn}{N}\right]$

$x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin \frac{2\pi kn}{N} + X_I(k) \cos \frac{2\pi kn}{N}\right]$

Deriving DFT from DTFT:

If $x(n)$ - aperiodic finite seq. with FT $X(\omega)$ Sampled at N equally spaces $\omega_K = 2\pi K/N$, $K = 0$ to $N-1$.

$$X(K) = X(\omega)\Big|_{\omega = 2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N},$$

DFT coefficients $\quad x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN).$

Finite during sequence $\quad \hat{x}(n) = \underline{x_p(n)} \quad 0 \le n \le N-1$

For $L \le N$, $\quad x(n) = \hat{x}(n). \quad 0 \le n \le N-1.$

(ie) $\quad X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$

Use of DFT in Linear Filtering:

Let, finite-duration sequence $x(n)$ of length L which excites an FIR filter of length M.

$x(n) = 0 \quad n < 0$ & $n \ge L$, $\quad h(n) = 0$ for $n \le 0$, $n \ge M$

O/P $y(n)$, using convolution of $x(n)$ & $h(n)$ as

$$y(n) = \sum_{K=0}^{M-1} h(K)\, x(n-K). \rightarrow \text{Time domain.}$$

Duration of $y(n) \Rightarrow L+M-1$

$Y(\omega) = X(\omega)\, H(\omega) \rightarrow$ frequency domain.

If $\quad Y(K) = Y(\omega)\Big|_{\omega = 2\pi K/N} \quad K = 0$ to $N-1$

$\quad = X(\omega)\, H(\omega)\Big|_{\omega = 2\pi K/N} \quad K = 0$ to $N-1$.

then $\quad Y(K) = X(K)\, H(K).$

N_1 point DFT is $X_1(k) = \sum_{n=0}^{N_1-1} x_1(n) e^{-j2\pi nk/N_1}$, $0 \le k \le N_1 - 1$

N_1 point DFT of $x_2(n)$ is $X_2(k) = \sum_{n=0}^{N_1-1} x_2(n) e^{-j2\pi nk/N_1}$, $0 \le k \le N_1 - 1$

If $DFT[x_1(n)] = X_1(k)$ and $DFT[x_2(n)] = X_2(k)$

then, $DFT[a x_1(n) + b x_2(n)] = a X_1(k) + b X_2(k)$.

3) circular shift of a sequence:

The shifted version of $x(n)$ i.e) shift $k = N$ as

$$x((n-N))_N = x(n).$$

$$x((n-m))_N = x(N-m+n).$$

If $DFT[x(n)] = X(k)$ then $DFT[x((n-m))_N] = e^{-j2\pi km/N} X(k)$

<u>Proof:</u>

$$DFT[x((n-m))]_N = \sum_{n=0}^{N-1} x((n-m))_N e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{m-1} x((n-m))_N e^{-j2\pi kn/N} + \sum_{n=m}^{N-1} x((n-m))_N e^{-j2\pi kn/N}.$$

Since, $x((n-m))_N = x(N-m+n)$, then

$$\sum_{n=0}^{m-1} x((n-m))_N e^{-j2\pi kn/N} = \sum_{n=0}^{m-1} x(N-m+n) e^{-j2\pi kn/N}$$

Let $N-m+n = l$, then

$$= \sum_{l=N-m}^{N-1} x(l) e^{-j2\pi k(-N+m+l)/N}$$

$$= \sum_{l=N-m}^{N-1} x(l) e^{-j2\pi k(l+m)/N}.$$

$e^{j2\pi k} = 1$ for $k = 0, 1, 2 \ldots$

Similarly,

$$\sum_{n=m}^{N-1} x((n-m))_N e^{-j2\pi kn/N} = \sum_{l=0}^{N-1-m} x(l) e^{-j2\pi k(m+l)/N}$$

$$DFT[x((n-m))_N] = \sum_{l=N-m}^{N-1} x(l)\, e^{-j2\pi k(m+l)/N} + \sum_{l=0}^{N-m-1} x(l)\, e^{-j2\pi k(m+l)/N}$$

$$= e^{-j2\pi km/N} \sum_{l=0}^{N-1} x(l)\, e^{-j2\pi kl/N}$$

$$= e^{-j2\pi km/N} \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}.$$

$$= e^{-j2\pi km/N}\, X(K).$$

4) Time reversal of a sequence:

Time reversal of N-pt seq. $x(n)$ is attained by wrapping the sequence $x(n)$ around the circle in clockwise direction. (ie) $x((-n))_N$.

$$x((-n))_N = x(N-n) \qquad 0 \le n \le N-1.$$

If $DFT[x(n)] = X(K)$ then $DFT[x((-n))_N] = X((-k))_N$
$= X(N-k)$.

Proof:

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n)\, e^{-j2\pi kn/N}$$

Changing the index n to $m = N-n$, we get

$$DFT[x(N-m)] = \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{j2\pi km/N} = \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi m(N-k)/N}$$

$$= X(N-k)$$

5) Circular Frequency Shift:

If $DFT[x(n)] = X(K)$ then $DFT[x(n)\, e^{j2\pi ln/N}] = X((K-l))_N$

Proof: $$DFT[x(n)\, e^{j2\pi ln/N}] = \sum_{n=0}^{N-1} x(n)\, e^{j2\pi ln/N}\, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(K-l)/N} = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(N+k-l)/N}$$

$$= X(N+k-l) = X((K-l))_N.$$

6) complex conjugate property:

If $DFT[x(n)] = X(K)$ then

$$DFT[x^*(n)] = X^*(N-K) = X^*((-K))_N.$$

<u>Proof</u>: $DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n)\, e^{-j2\pi kn/N}$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(N-K)/N}\right]^* = X^*(N-K)$$

$$DFT[x^*(N-n)] = X^*(K).$$

$$IDFT[X^*(K)] = \frac{1}{N}\sum_{k=0}^{N-1} X^*(K)\, e^{j2\pi kn/N}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(K)\, e^{-j2\pi kn/N}\right]^* = \frac{1}{N}\left[\sum_{K=0}^{N-1} X(K)\, e^{j2\pi k(N-n)/N}\right]^*$$

$$= x^*(N-n).$$

$\therefore DFT[x^*(N-n)] = X^*(K).$

7) <u>Circular</u> <u>Convolution</u>:

Let $x_1(n)$ & $x_2(n)$ are finite duration sequences both of length N with DFTs $X_1(K)$ & $X_2(K)$, then

$$X_3(K) = X_1(K)\, X_2(K).$$

$$x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m)\, x_{2p}(n-m). \quad (or)$$

$$x_3((n))_N = \sum_{m=0}^{N-1} x_1((m))_N\, x_2((n-m))_N$$

For $0 \le n \le N-1$; $x_3((n))_N = x_3(n)$. ; $x_1((m))_N = x_1(m)$.

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$$x_3(n) = x_1(n) \; \textcircled{N} \; x_2(n).$$

$$DFT[x_1(n) \; \textcircled{N} \; x_2(n)] = X_1(K)\, X_2(K).$$

8) Circular Correlation:

For complex-valued sequences $x(n)$ & $y(n)$, if

$DFT[x(n)] = X(K)$ & $DFT[y(n)] = Y(K)$, then

$$DFT[\bar{\gamma}_{xy}(l)] = DFT\left[\sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N\right] = X(K)\, Y^*(K)$$

9) Multiplication of two sequences:

If $DFT[x_1(n)] = X_1(k)$ & $DFT[x_2(n)] = X_2(k)$ then

$$DFT[x_1(n)\, x_2(n)] = \frac{1}{N}\left[X_1(K) \; (N) \; X_2(K)\right]$$

10) Parseval's Theorem:

If $DFT[x(n)] = X(K)$ & $DFT[y(n)] = Y(K)$ then

$$\leftarrow \sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, Y^*(K).$$

$$LHS \Leftarrow \sum_{n=0}^{N-1} x(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} Y^*(K)e^{-j2\pi kn/N}\right] \Leftarrow \frac{1}{N}\sum_{K=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}}\right] Y^*(K) \Leftarrow RHS$$

$$LHS \sum_{n=0} x(n)\left[\frac{1}{N}\sum_{K=0} X^*(k)e^{-j2\pi kn/N}\right] = \frac{1}{N}\sum_{k=0} X(k)\, Y^*(k)$$

Problem: Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and find the IDFT of $Y(K) = \{1, 0, 1, 0\}$

Assume $N = L = 4$.

$$X(K) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \quad K = 0, 1 \ldots N-1$$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3) = 1+1+0+0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi n/2} = x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$

$$= 1 + \cos\pi/2 - j\sin\pi/2 = 1 - j.$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = 1 + \cos\pi - j\sin\pi = 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j3n\pi/2} = 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = 1 + j.$$

$$X(K) = \{2, 1-j, 0, 1+j\}$$

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{j2\pi kn/N}, \quad n=0,1\cdots N-1$$

$$y(0) = \frac{1}{4}\sum_{k=0}^{3} Y(k), \quad n=0,1,2,3$$

$$= \frac{1}{4}\left[Y(0)+Y(1)+Y(2)+Y(3)\right] = \frac{1}{4}\left[1+0+1+0\right] = 0.5$$

$$y(1) = \frac{1}{N}\sum_{k=0}^{3} Y(k)\, e^{j\pi k/2}$$

$$= \frac{1}{4}\left[1+0+\cos\pi + j\sin\pi + 0\right] = 0$$

$$y(2) = \frac{1}{4}\left[1+0+\cos 2\pi + j\sin 2\pi + 0\right] = 0.5$$

$$y(3) = \frac{1}{4}\left[1+0+\cos 3\pi + j\sin 3\pi + 0\right] = 0$$

$$\therefore y(n) = \{0.5, 0, 0.5, 0\}$$

Circular Convolution:

1) Concentric circle method   2) Matrix multiplication method.

1) Concentric Circle method:

Given two sequences $x_1(n)$ & $x_2(n)$, then circular convolution of two sequences $x_3(n) = x_1(n) \textcircled{N} x_2(n)$

<u>Steps</u>: 1) Graph N samples of $x_1(n)$ as equally spaced points around an outer circle in counter clockwise.

2) Start at the same point as $x_1(n)$ graph N samples of $x_2(n)$ as equally spaced points around an inner circle in clockwise direction.

3) Multiply corresponding samples on two circles and sum the products to produce o/p.

4) Rotate the inner circle one sample at a time in counter clockwise and goto step 3 to obtain next value of o/p

5) Repeat step 4 until inner circle first sample lines up

2) Matrix Multiplication Method:

Circular convolution of two sequences canbe obtained by representing sequences in matrix form as below

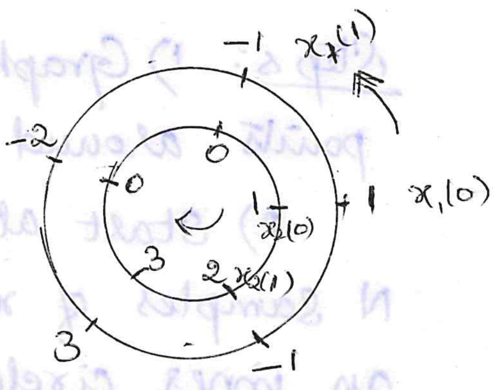
$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(2) \\ \vdots & \vdots & & & \vdots \\ x_2(N-2) & x_2(N-3) & & & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & & & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ \vdots \\ x_1(N-2) \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix}$$

The sequence $x_2(n)$ is repeated via circular shift of samples and represented in $N \times N$ matrix form. The sequence $x_1(n)$ is represented as column matrix. The multiplication of these two matrices gives the sequences $x_3(n)$.

Problem: 1) Find the circular convolution of two finite duration sequences $x_1(n) = \{1, -1, -2, 3, -1\}$, $x_2(n) = \{1, 2, 3$

Soln: To find circular convolution, both sequences must be same length. So, append two zeros in $x_2(n)$.

$$x_1(n) = \{1, -1, -2, 3, -1\}$$
$$x_2(n) = \{1, 2, 3, 0, 0\}$$

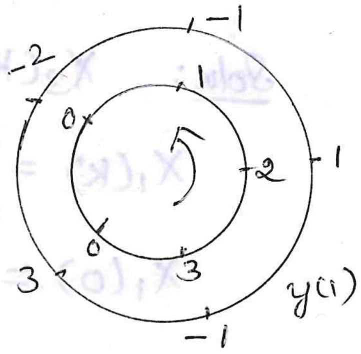


Graph $x_1(n)$ on outer circle in anti clockwise & $x_2(n)$ on the inner circle which starts at same point as $x_1(n)$ in clockwise direction.

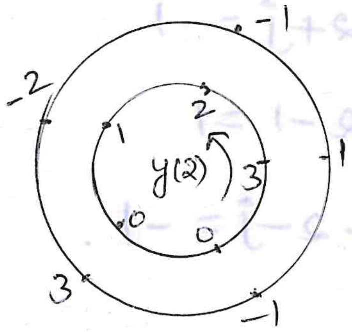
Multiply the corresponding samples and add to obtain $y(0) = 1(1) + 0(-1) + 0(-2) + 3(3) + 2(-1)$
$= 8$

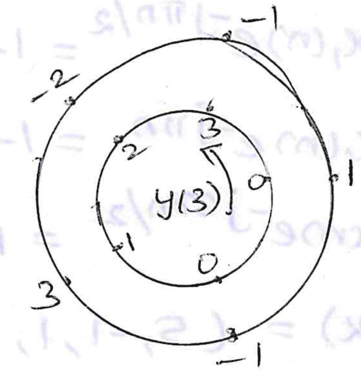
Rotate the inner circle in counterclockwise by one sample, multiply corresponding samples to obtain

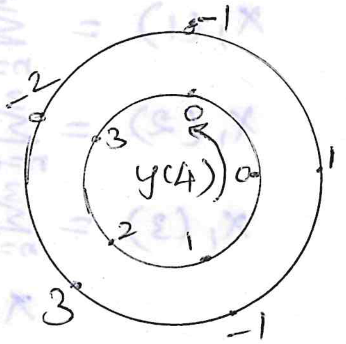
$y(1) = 1(2) + (-1)1 + (-2)0 + 3(0) + 3(-1)$
$\quad = -2$



$y(2) = 3(1) + 2(-1) + 1(-2) + 0(3) - 1(0)$
$\quad = -1$







Obtain the remaining samples by repeating above steps until the inner circle first sample lines up with the first sample of exterior circle.

$y(3) = 0(1) + 3(-1) + 2(-2) + 1(3) + (-1)0 = -4$

$y(4) = 0(1) + (0)(-1) + 3(-2) + 3(2) + 1(-1) = -1$

$y(n) = \{8, -2, -1, -4, -1\}$

Matrix Method:

$x_1(n) = \{1, -1, -2, 3, -1\} \quad x_2(n) = \{1, 2, 3, 0, 0\}$

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}; \quad \underline{\text{Ans:}}\ y(n) = \{8, -2, -1, -4, -1\}$$

2) Perform the circular convolution of the following sequences $x_1(n) = \{1, 1, 2, 1\}$, $x_2(n) = \{1, 2, 3, 4\}$ using DFT & IDFT method

Soln: $X_3(K) = X_1(K) \cdot X_2(K)$

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N}, \quad K = 0, 1 \cdots N-1, \quad N = 4.$$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) = 1+1+2+1 = 5$$

$$X_1(1) = \sum_{n=0}^{3} x_1(n) e^{-j\pi n/2} = 1-j-2+j = -1$$

$$X_1(2) = \sum_{n=0}^{3} x_1(n) e^{-j\pi n} = 1-1+2-1 = 1$$

$$X_1(3) = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2} = 1+j-2-j = -1.$$

$$X_1(K) = (5, -1, 1, -1).$$

Similarly, $X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi nk/N}$, $k = 0$ to $N-1$.

$X_2(0) = 1+2+3+4 = 10$, $X_2(1) = 1+2(-j)+3(-1)+4(j) = -2+j2$

$X_2(2) = 1+2(-1)+3(1)+4(-1) = -2$, $X_2(3) = -2-j2$.

$$X_2(K) = (10, -2+j2, -2, -2-j2).$$

$\therefore X_3(K) = X_1(K) \cdot X_2(K) = \{50, 2-j2, -2, 2+j2\}$

$$x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(K) e^{j2\pi nk/N}, \quad n = 0, 1 \cdots N-1.$$

$$x_3(0) = \frac{1}{4} \sum_{k=0}^{3} X_3(K) = \frac{1}{4}(50+2-j2-2+2+j2) = 13$$

$$x_3(1) = \frac{1}{4}[50+(2-j2)j+(-2)(-1)+(2+j2)(-j)] = 14$$

$$x_3(2) = \frac{1}{4}[50+(2-j2)(-1)+(-2)(1)+(2+j2)(-1)] = 11$$

$$x_3(3) = \frac{1}{4}[50+(2-j2)(-j)+(-2)(-1)+(2+j2)(j)] = 12$$

$\therefore x_3(n) = \{13, 14, 11, 12\}$

Linear Filtering using DFT:

Filtering methods based on DFT:

Suppose an input sequence $x(n)$ of long duration then it is divided into blocks. The successive blocks are processed seperately one at a time & the result are combined to yield the desired output sequence.

Filtering Long data Sequences:

Methods: 1) Overlap - Save Method

Let length of an i/p sequence Ls.
Length of impulse response is M.
I/p sequence is divided into blocks of data of size ~~L = desired~~ req. $[N = L+M-1.]$

* Each block consists of last $(M-1)$ data points of previous block followed by L new data points.
* For first block of data, the first $M-1$ points are set to zero.

$$x_1(n) = \{\underbrace{0, 0, \ldots 0}_{(M-1)\text{ zeros}}, x(0), x(n) \ldots x(L-1)\}$$

$$x_2(n) = \{\underbrace{x(L-M+1), \ldots x(L-1)}_{\text{Last }(M-1)\text{ points from }x_1(n)}, \underbrace{x(L) \ldots x(2L-1)}_{L\text{ new points}}$$

$$x_3(n) = \{\underbrace{x(2L-M+1), \ldots x(2L-1)}_{\text{Last }(M-1)\text{ points from }x_2(n)}, \underbrace{x(2L) \ldots x(3L-1)}_{L\text{ new datapoints}}\}$$

and so on.

Impulse response of FIR filter is increased in length by appending $L-1$ zeros and N-point circular convolution of $x_i(n)$ and $h(n)$ is computed.

(ie) $y_i(n) = x_i(n)$ Ⓝ $h(n)$.

Discard the first $(M-1)$ points. Remaining points construct final result.

2) Overlap - add method:

Let length of the sequence is L_s.

Length of impulse response is M.

Sequence is divided into blocks of data size having length L and $M-1$ zeros are appended to it to make the data size $L+M-1$.

Data blocks: $x_1(n) = \{x(0), x(1) \cdots x(L-1), \underbrace{0, 0 \cdots}_{(M-1)\text{ zeros append}}\}$

$x_2(n) = \{x(L), x(L+1) \cdots x(2L-1), 0, 0 \cdots\}$

$x_3(n) = \{x(2L), x(2L+1) \cdots x(3L-1), \underbrace{0, 0 \cdots}_{(M-1)\text{ zeros appended}}\}$

Now, $(L-1)$ zeros are added to the impulse response and N-point circular convolution is performed. Each data block is terminated with $M-1$ zeros, last $M-1$ points from each o/p block must be overlapped and added to the first $M-1$ points of the succeeding block. So, it is called overlap-add method.

$\therefore y(n) = y_1(0), y_1(1) \cdots y_1(L) + y_2(0), y_2(N-1) + y_2(M-2)$
$y_2(M) \cdots y_2(L) + y_3(0), y_2(L+1) + y_3(1) \cdots y_3(N-1)\}$

Example: Find the o/p $y(n)$ of a filter whose impulse response is $h(n) = \{1,1,1\}$ & i/p $x(n) = \{3,-1,0,1,3,2,0,1,2,1\}$ using overlap-save & overlap-add methods.

i) Overlap-save method:

$x_1(n) = \{\underbrace{0, 0}_{M-1=2\text{ zeros}}, \underbrace{3, -1, 0}_{L\text{ points}=3}\}$ $\quad x_3(n) = \{3, 2, 0, 1, 2\}$

$x_2(n) = \{\underbrace{-1, 0}_{\text{previous data}}, 1, 3, 2\}$ $\quad x_4(n) = \{1, 2, 1, 0, 0\}$

Given $h(n) = \{1, 1, 1\}$

Increase length to $\overset{2M-1}{\cancel{L+M-1}} = 5$ by adding two zeros

$h(n) = \{1, 1, 1, 0, 0\}$

$y_1(n) = x_1(n) \text{ (N) } h(n) = \{\underbrace{-1, 0}, 3, 2, 2\}$

$y_2(n) = x_2(n) \text{ (N) } h(n) = \{\underbrace{4, 1}, 0, 4, 6\}$

$y_3(n) = x_3(n) \text{ (N) } h(n) = \{\underbrace{6, 7}, 5, 3, 3\}$

$y_4(n) = x_4(n) \text{ (N) } h(n) = \{\underbrace{1, 3}_{\text{discard}}, 4, 3, 1\}$

$\therefore\ y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

ii) Overlap-Add method:

Let length of data block be 3. Two zeros are added to bring the length to $L+M-1 = 5$

$x_1(n) = \{3, -1, 0, 0, 0\}$, $x_2(n) = \{1, 3, 2, 0, 0\}$

$x_3(n) = \{0, 1, 2, 0, 0\}$, $x_4(n) = \{1, 0, 0, 0, 0\}$

$y_1(n) = x_1(n) \text{ (N) } h(n) = \{3, 2, 2, -1, 0\}$

$y_2(n) = x_2(n) \text{ (N) } h(n) = \{1, 4, 6, 5, 2\}$

$y_3(n) = x_3(n) \text{ (N) } h(n) = \{0, 1, 3, 3, 2\}$

$y_4(n) = x_4(n) \text{ (N) } h(n) = \{1, 1, 1, 0, 0\}$

```
3  2  2  -1  0   Add
          1  4  6, 5  2
                   0, 1, 3, 3, 2
                            1, 1, 1, 0, 0
```

$\therefore\ y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

Fast Computation of DFT

The Fast Fourier Transform:

- Highly efficient procedure for computing DFT of a finite series and requires less no. of computations.

- FFT algorithms exploit symmetry & periodicity properties of twiddle factor W_N^k & reduces no. of complex multiplications required to perform DFT from N^2 to $\frac{N}{2}\log_2 N$.

- FFT algorithms are based on principle of decomposing the computation of DFT of a sequence of length N into successively smaller DFTs.

Algorithms: 1) Decimation in-time
2) Decimation in-frequency

Radix - 2

Decimation - in - time Algorithm FFT:

- Radix-2 DIT FFT algorithm which means no. of o/p points N is expressed as power of 2.

(ie) $N = 2^M$, M - integer.

Let $x(n)$ - N point sequence - Decimate into two sequences of equal length N/2. One sequence consists of even indexed & other of odd indexed values.

(ie) $x_e(n) = x(2n)$, $n = 0, 1 \dots N/2 - 1$

$x_o(n) = x(2n+1)$, $n = 0, 1 \dots \frac{N}{2} - 1$

N-point DFT of $x(n)$ is written as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1 \dots N-1$$

Seperating $x(n)$ into even & odd indexed values,

$$X(K) = \sum_{\substack{n=0 \\ (\text{even})}}^{N-1} x(n)\, W_N^{nk} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x(n)\, W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_N^{2nk}.$$

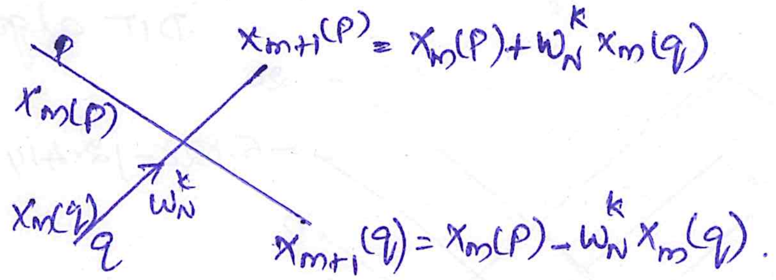
$$X(K) = \sum_{n=0}^{N/2-1} x_e(n)\, W_N^{2nk} + W_N^{k} \sum_{n=0}^{N/2-1} x_o(n)\, W_N^{2nk}.$$

But $W_N^2 = (e^{-j2\pi/N})^2 = e^{-j2\pi/N/2} = W_{N/2}$.

$$\therefore X(K) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_e(n)\, W_{N/2}^{nk}}_{N/2 \text{ point DFT of even indexed seq.}} + W_N^{k} \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_o(n)\, W_{N/2}^{nk}}_{N/2 \text{ point DFT of odd indexed sequence.}}$$

$$X(K) = X_e(k) + W_N^{K} X_o(k).$$

Flow Graph:



In DIT algorithm, o/p sequence is in natural order.
I/p sequence is stored in shuffle order.

Bit-reversal

(8-bit)

I/p	Binary	Bit revers	Bit rever sample
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

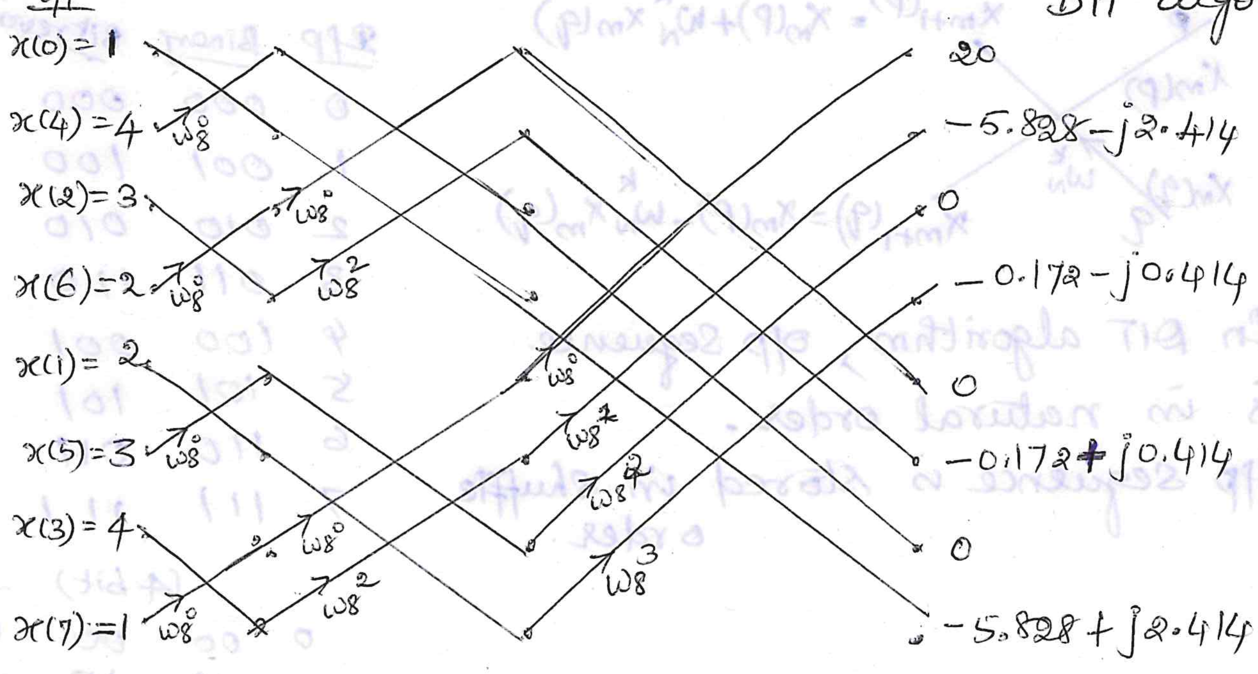
(4 bit)

0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3

Steps of radix-2 DIT-FFT algorithm:

1) The no. of i/p samples $N = 2^M$ where M - integer.
2) The i/p sequence is shifted through bit reversal.
3) The no. of stages is $M = \log_2 N$.
4) Each stage consists of $N/2$ butterflies.
5) Inputs / Outputs are seperated by 2^{m-1} samples in each butterfly, where m - stage index (ie) for first stage m=1.
6) No. of complex multiplications $\frac{N}{2}\log_2 N$.
7) No. of complex additions $N \log_2 N$.
8) Twiddle factor exponents are function of stage index m and is given by $K = \frac{Nt}{2^m}$. $t = 0, 1, \cdots 2^{m-1}-1$
9) No. of sets of butterflies in each stage is 2^{M-m}.
10) The exponent repeat factor (ERF), which is no. of times the exponent sequence associated with m is repeated is given by 2^{M-m}.

ex: 1) Find the DFT of a seq. $x(n) = \{1,2,3,4,4,3,2,1\}$ using DIT algorithm.

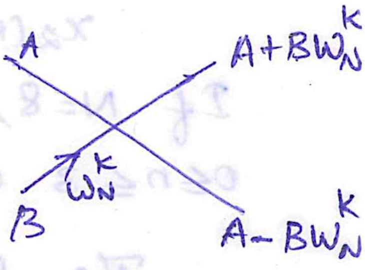


The twiddle factors associated with the flowgraph are

$W_8^0 = 1$, $W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$

$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$

$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j\frac{3\pi}{4}} = -0.707 - j0.707$

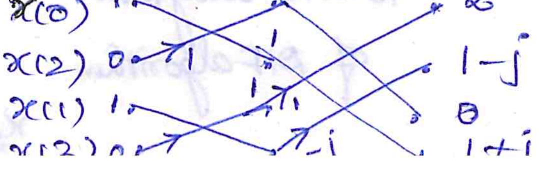


I/P	O/P of Stage 1	O/P of Stage 2	Output
1	$1+4=5$	$5+5=10$	$10+10=20$
4	$1-4=-3$	$-3+(-j)1=-3-j$	$-3-j+(0.707-j0.707)(-1-3j) = -5.828-j2.414$
3	$3+2=5$	$5-5=0$	0
2	$3-2=1$	$-3-(-j)1=-3+j$	$(-3+j)+(-0.707-j0.707)(-1+3j) = -0.172-j0.414$
2	$2+3=5$	$5+5=10$	$10-10=0$
3	$2-3=-1$	$-1+(-j)3=-1-3j$	$-3-j-(0.707-j0.707)(-1-3j) = -0.172+j0.414$
4	$4+1=5$	$5-5=0$	0
1	$4-1=3$	$-1-(-j)3=-1+3j$	$(-3+j)-(-0.707-j0.707)(-1+3j) = -5.828+j2.414$

$\therefore X(k) = \{20, -5.828-j2.414, 0, -0.172-j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414\}$

2) Find 4 pt DFT of seq. $x(n) = \{1,1,0,0\}$ using DIT-FFT

1) $N=4$
2) $M = \log_2 N = 2$ (stages)
3) $N/2 = 2$ (Butterflies)
4) $k = \frac{Nt}{2^m}$, $t = 0,1 \ldots 2^{m-1}-1$.



Decimation - in - Frequency Algorithm FFT:

In DIF algorithm, the o/p Sequence $X(k)$ is divided into smaller sub sequences. I/p sequence $x(n)$ is partitioned into two sequences each of length $\frac{N}{2}$ samples.

$$x_1(n) = x(n), \quad n = 0,1,2 \dots N/2 - 1$$

$$x_2(n) = x(n+N/2), \quad n = 0,1,2 \dots N/2 - 1.$$

If $N=8$, the first sequence $x_1(n)$ has values for $0 \le n \le 3$ and $x_2(n)$ has values for $4 \le n \le 7$.

The N point DFT can be written as

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{nk}.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{(n+\frac{N}{2})k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + W_N^{Nk/2} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{nk}.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + e^{-j\pi k} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{nk}.$$

When k is even $e^{-j\pi k} = 1$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] W_N^{2nk} = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] W_{N/2}^{nk}.$$

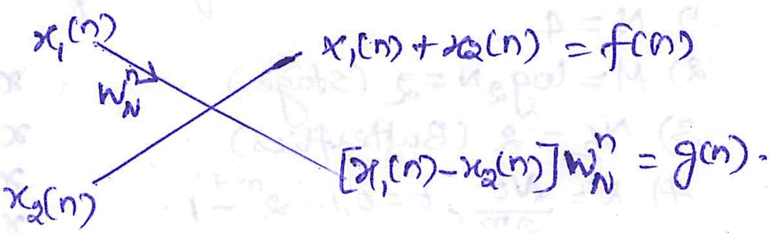
where $f(n) = x_1(n) + x_2(n)$. $\quad = \sum_{n=0}^{\frac{N}{2}-1} f(n) W_{N/2}^{nk}.$

When k is odd, $e^{-j\pi k} = -1$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)] W_N^{(2k+1)n} = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)] W_N^{n} W_{N/2}^{n}$$

where $g(n) = [x_1(n) - x_2(n)] W_N^{n}$. $\quad = \sum_{n=0}^{\frac{N}{2}-1} g(n) W_{N/2}^{nk}.$

Basic diagram of DIF algorithm

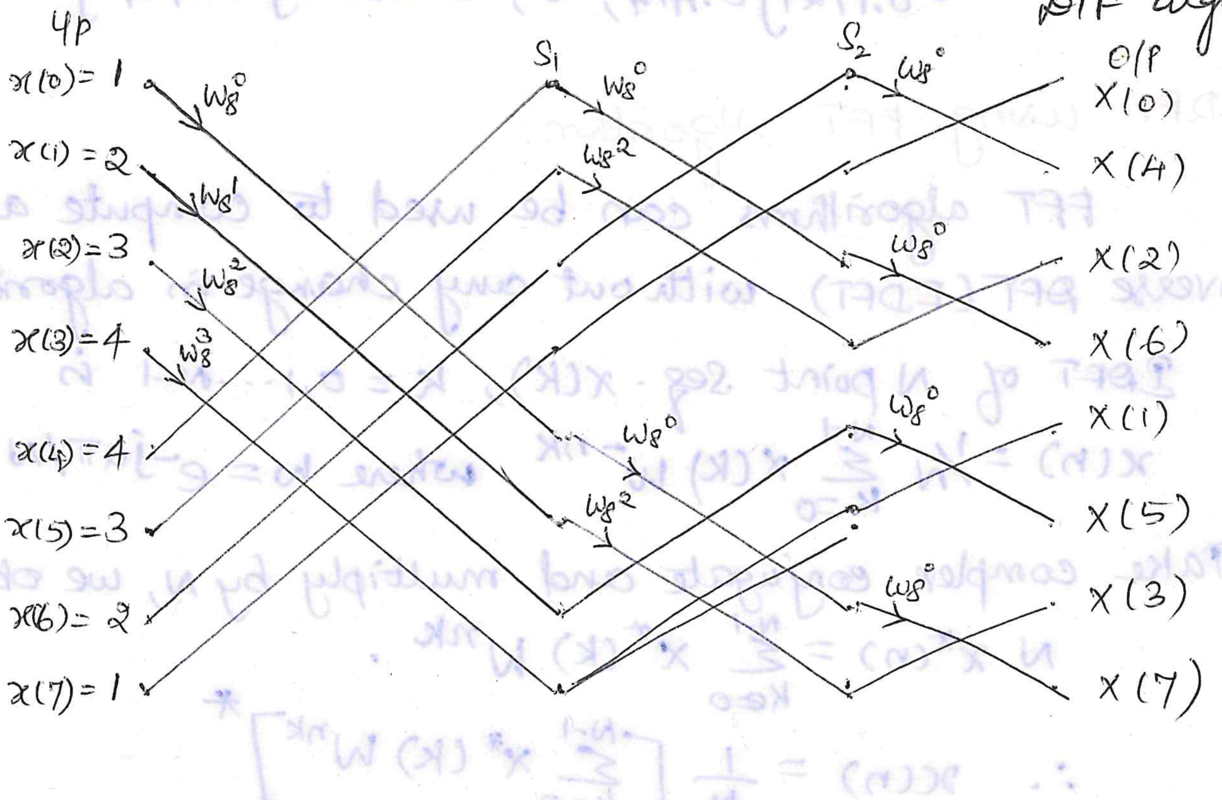


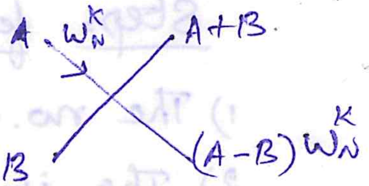
Steps for Radix-2 DIF-FFT Algorithm:

1) The no. of i/p samples $N = 2^M$, where M - no. of stages
2) The i/p sequence is in natural order.
3) The no. of stages in flow graph is $M = \log_2 N$.
4) Each stage consists of $\frac{N}{2}$ butterflies.
5) Inputs/Outputs for each butterfly are seperated by 2^{M-m} samples, where m - stage index.
6) The no. of complex multiplications is $\frac{N}{2}\log_2 N$.
7) The no. of complex additions is $N\log_2 N$.
8) The twiddle factor exponents are a function of stage index m and is $k = \frac{Nt}{2^{M-m+1}}$, $t = 0, 1 \cdots 2^{M-m} - 1$
9) The no. of sets of butterflies in each stage is 2^{m-1}.
10) The exponent repeat factor (ERF), which is no. of times the exponent seq. associated with m repeated is 2^{m-1}.

Speed factor $= \frac{N^2}{\frac{N}{2}\log_2 N}$

Ex: Find DFT of a sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIF algorithm.





I/P	S_1	S_2	O/P
1	$1+4=5$	$5+5=10$	$10+10=20$
2	$2+3=5$	$5+5=10$	$10-10=0$
3	$3+2=5$	$5-5=0$	0
4	$4+1=5$	$(5-5)W_8^2=0$	0
4	$(1-4)W_8^0=-3$	$-3+(-j)=-3-j$	$-3-j-2.828-j1.414$ $=-5.828-j2.414$
3	$(2-3)(0.707-j0.707)$ $=-0.707+j0.707$	$-0.707+j0.707$ $+(-2.121-j2.121)$ $=-2.828-j1.414$	$-3-j+2.828+j1.414$ $=-0.172+j0.414$
2	$(3-2)(-j)=-j$	$-3-(-j)=-3+j$	$-3+j+2.828-j1.414$ $=-0.172-j0.414$
1	$(4-1)(-0.707-j0.707)$ $=-2.121-j2.121$	$(-0.707+j0.707+2.121$ $+j2.121)(-j)$ $=2.828-j1.414$	$-3+j-2.828+j1.414$ $=-5.828+j2.414$

$$X(K)=\{20,\ -5.828-j2.414,\ 0,\ -0.172-j0.414,\ 0,\ -0.172+j0.414,\ 0,\ -5.828+j2.414\}$$

IDFT using FFT Algorithm:

FFT algorithms can be used to compute an Inverse DFT (IDFT) without any change in algorithm.

IDFT of N point seq. $X(K)$, $k=0,1\ldots N-1$ is

$$x(n)=\frac{1}{N}\sum_{K=0}^{N-1}X(K)\,W^{-nk} \quad \text{where } W=e^{-j2\pi/N}.$$

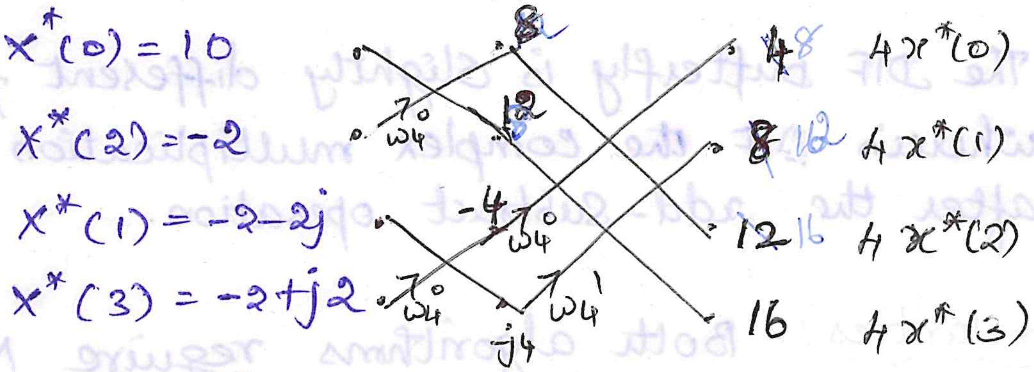
Take complex conjugate and multiply by N, we obtain

$$N\,x^*(n)=\sum_{K=0}^{N-1}X^*(K)\,W^{nk}.$$

$$\therefore\ x(n)=\frac{1}{N}\left[\sum_{K=0}^{N-1}X^*(K)\,W^{nk}\right]^*$$

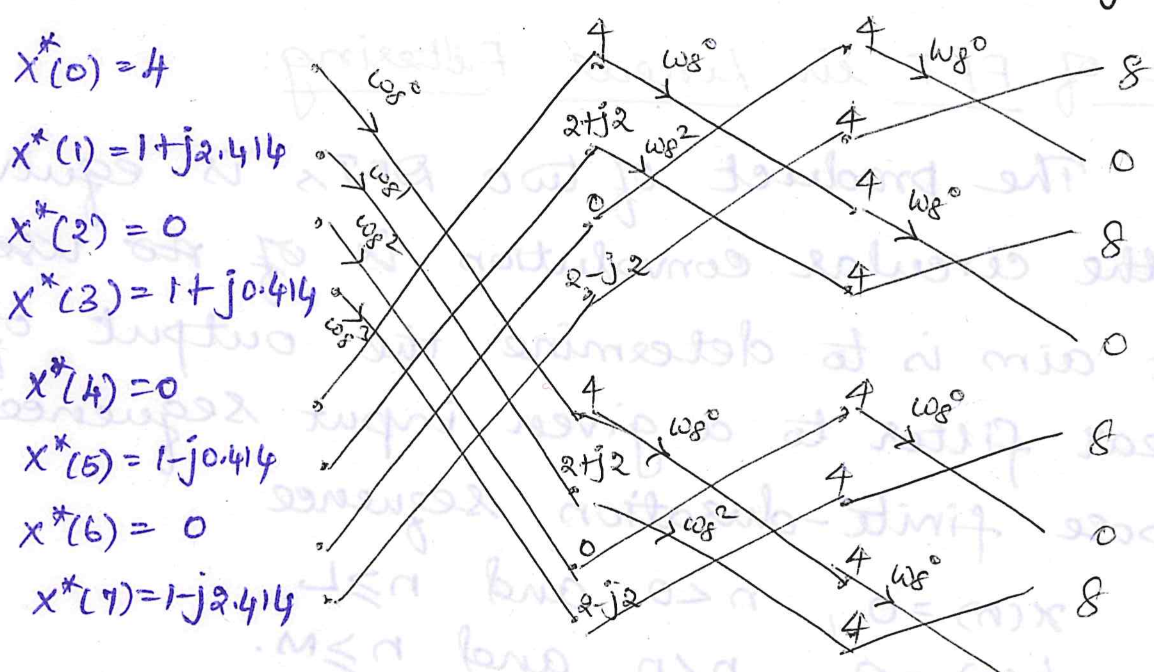
1) Find IDFT of the sequence $X(K) = \{10, -2+j2, -2, -2-j2\}$ using DIT algorithm.

Soln: Twiddle factors $W_4^0 = 1$, $W_4^1 = -j$.



$x(n) = \{1, 2, 3, 4\}$ Ans.

2) Find IDFT of the sequence $X(K) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$ using DIF algorithm



The o/p $8x^*(n)$ is in reversal order.

$\therefore x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$ Ans

Differences between DIT & DIF:

1) In DIT, i/p is bit reversed & o/p is in natural order.
In DIF, i/p is in natural order & o/p is in bit reversed order.

2) The DIF butterfly is slightly different from DIT wherein DIF the complex multiplication takesplace after the add-subtract operation.

Similarities: Both algorithms require $N \log_2 N$ operations to compute DFT. Both are done in-place [I/p & o/p stored in same location] and both need to perform bit reversal at some place during computation.

Use of FFT in Linear Filtering:

The product of two DFTs is equivalent to the circular convolution is of ~~no~~ ~~use~~ to ~~the~~ but aim is to determine the output of a linear filter to a given input sequence.

Suppose finite-duration sequence

$x(n) = 0, \quad n < 0$ and $n \geq L$

$h(n) = 0, \quad n < 0$ and $n \geq M$.

$h(n)$ - impulse response of the FIR filter.

The o/p sequence $y(n)$ of the filter is in time domain as the convolution of $x(n)$ & $h(n)$. (ie) $y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$.

$$Y(\omega) = X(\omega)\, H(\omega)$$

If Sequence $y(n)$ is represented in the frequency domain by samples of its Spectrum $Y(\omega)$ at a Set of discrete frequencies, no. of distinct samples equal or exceed $L+M-1$.

DFT of Size $n \geq L+M-1$ is required to represent $\{y(n)\}$ in the frequency domain.

$$Y(K) = Y(\omega)\big|_{\omega = 2\pi k/N}, \quad K = 0, 1 \cdots N-1.$$

$$= X(\omega)\, H(\omega)\big|_{\omega = 2\pi k/N}, \quad K = 0, 1 \cdots N-1$$

$$Y(K) = X(K)\, H(K).$$

$X(K)$ & $h(K)$ are N-point DFTs of corresponding sequences $x(n)$ & $h(n)$

If overlap add method is used to perform linear filtering, method using FFT is same. Only difference is that N-point data blocks consist of L new data points and M-1 additional zeros. After IDFT is computed for each data block, N-point filtered blocks are overlapped and M-1 overlapping data points between Successive o/p records are added together.

computational complexity of FFT for linear filtering is, one time computation of $H(K)$ is insignificant.

Each FFT requires $(N/2)\log_2 N$ complex multiplication $N\log_2 N$ additions.

→ N complex multiplications and $N-1$ additions required to compute $Y_m(k)$.

→ $(N\log_2 2N)/L$ complex multiplications per o/p data point & $(2N\log_2 2N)/L$ additions per o/p data point. The overlap add method requires an incremental increase of $(M-1)/L$ in the no. of additions.

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UNIT II . IIR FILTER DESIGN

Introduction:

* Digital filter - Linear-time invariant discrete time System
* FIR filter - Finite Impulse Response filter
  - Non recursive type
  - o/p depends on present & previous i/p.
* IIR filter - Infinite Impulse Response Filter
  - Recursive type
  - o/p depends on present i/p, past i/p & o/p.

Impulse response $h(n)$ for realizable filter is

$$h(n) = 0 \quad \text{for } n \leq 0$$

For stability, $\sum_{n=0}^{\infty} |h(n)| < \infty$

Transfer function of IIR filter is

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}.$$

Structures of IIR:

1) Direct form-I: Consider LTI recursive system described by difference equation.

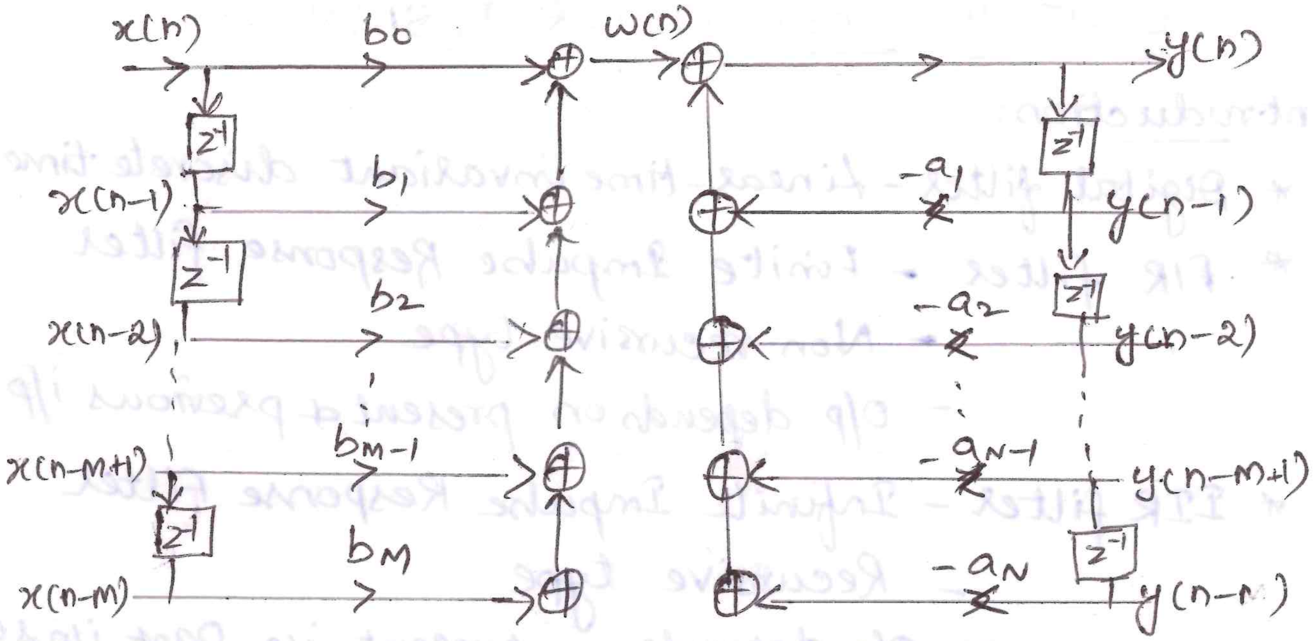
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$= -a_1 y(n-1) - a_2 y(n-2) \cdots - a_{N-1} y(n-N+1) - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M).$$

Let $b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) = w(n)$

then $y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) + w(n)$.

This realization requires $M+N+1$ multiplications, $M+N$ additions and $M+N+1$ memory locations.



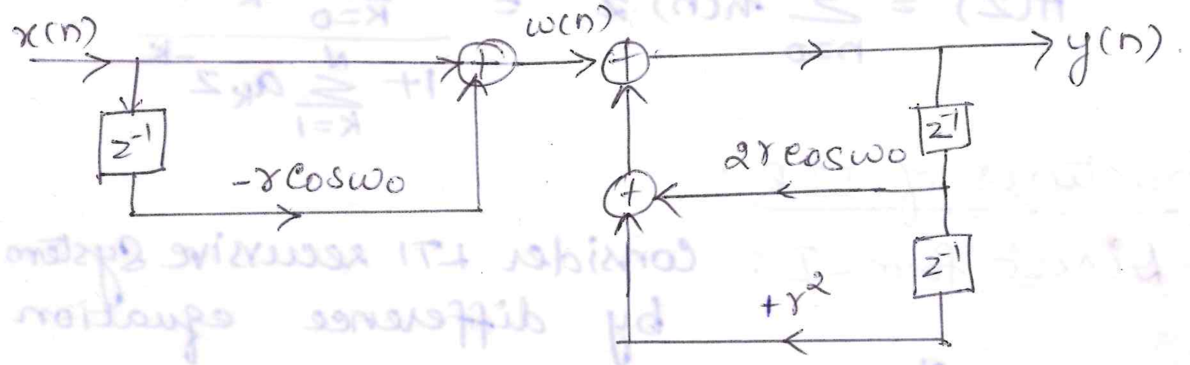
Direct form-I realization

Ex: Realize the second order digital filter

$$y(n) = 2\gamma \cos\omega_0\, y(n-1) + \gamma^2 y(n-2) + x(n) - \gamma\cos\omega_0\, x(n-1)$$

Soln: Let $x(n) - \gamma\cos\omega_0\, x(n-1) = w(n)$

then $y(n) = 2\gamma\cos\omega_0\, y(n-1) + \gamma^2 y(n-2) + w(n)$.



2) Direct form-II :

Consider the difference equation of the form

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

$\therefore$ System function $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$

Let $\dfrac{Y(z)}{X(z)} = \dfrac{Y(z)}{W(z)} \cdot \dfrac{W(z)}{X(z)}$

where $\frac{W(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$

which gives us $W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \cdots - a_N z^{-N} W(z)$

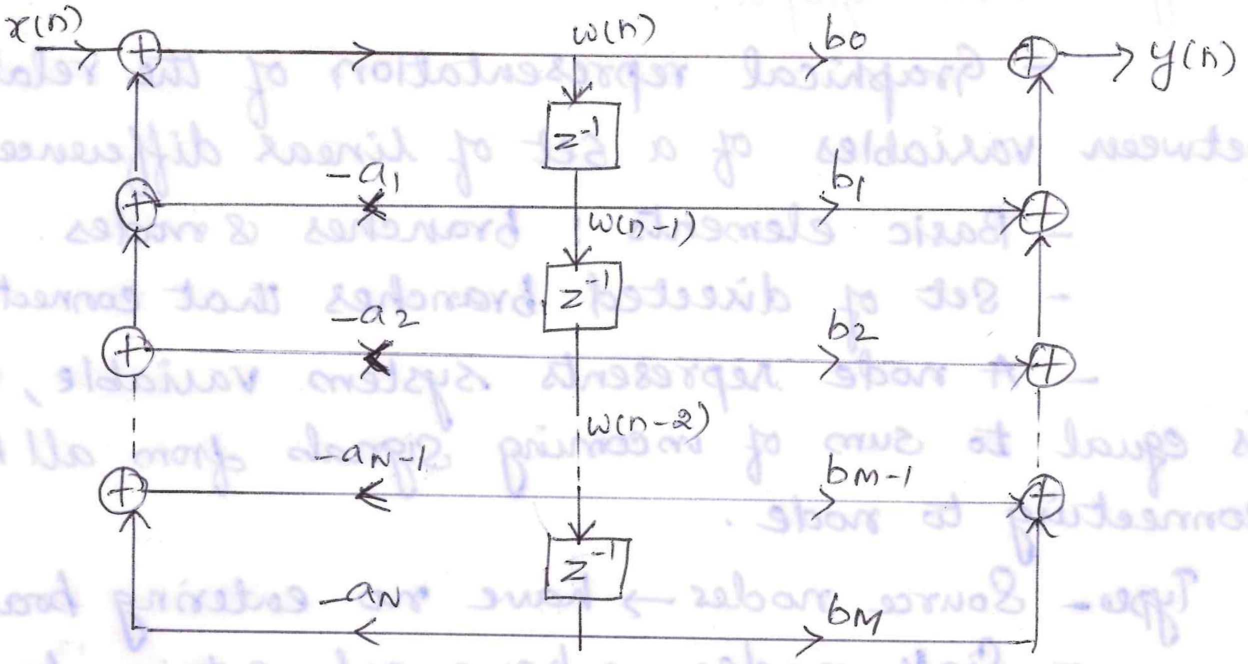
and $\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$ from which

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \cdots + b_M z^{-M} W(z)$$

(ie) $y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + \cdots + b_M w(n-M)$

But, $w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) \cdots - a_N w(n-N)$

This realization requires M+N+1 multiplications, M+N additions and max. of $\{M, N\}$ memory locations.



Direct Form - II Structure

Ex: Determine the direct form - II realization for the following system $y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$

Soln: System function $\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$

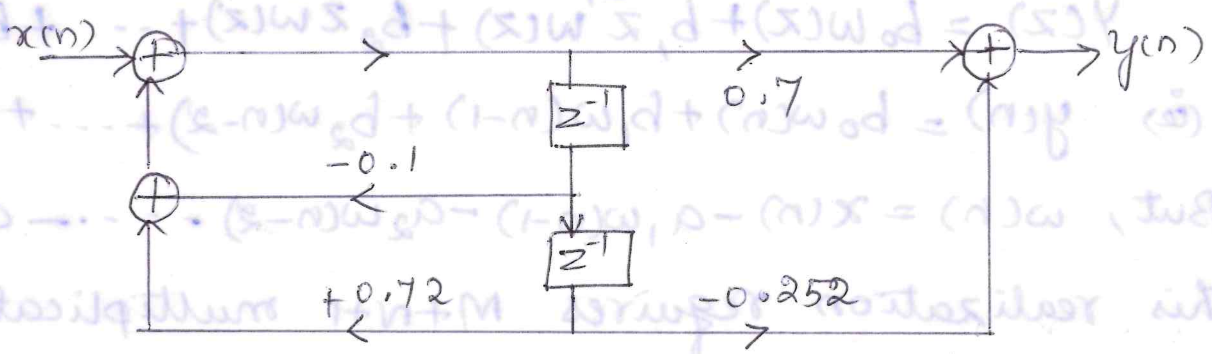
Let $\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-2}$; $Y(z) = 0.7W(z) - 0.252z^{-2}W(z)$

Then $y(n) = 0.7\,w(n) - 0.252\,w(n-2)$

Let $\dfrac{W(z)}{X(z)} = \dfrac{1}{1+0.1z^{-1}-0.72z^{-2}}$

$W(z) = X(z) - 0.1z^{-1}W(z) + 0.72\,z^{-2}W(z)$

$\therefore\ w(n) = x(n) - 0.1\,w(n-1) + 0.72\,w(n-2)$.

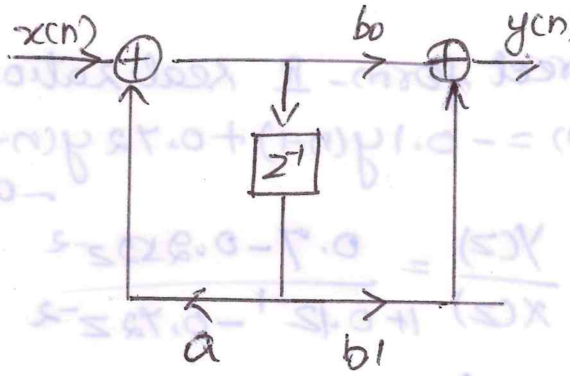


3) Signal Flow Graph:

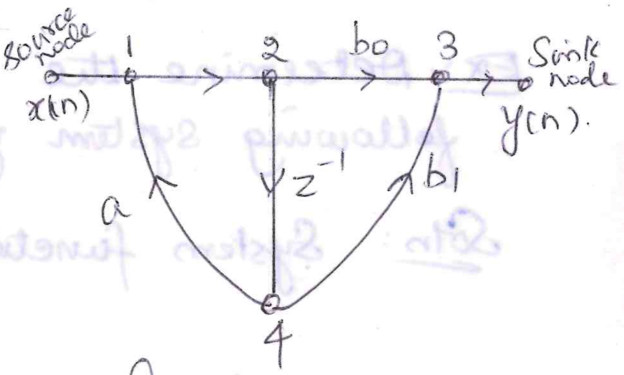
- Graphical representation of the relationship between variables of a set of linear difference equation
- Basic elements : branches & nodes.
- Set of directed branches that connect at nodes
- A node represents system variable, which is equal to sum of incoming signals from all branches connecting to node.

Types - Source nodes → have no entering branches
- Sink nodes → have only entering branches

ex:



Block diagram



Signal Flow Graph

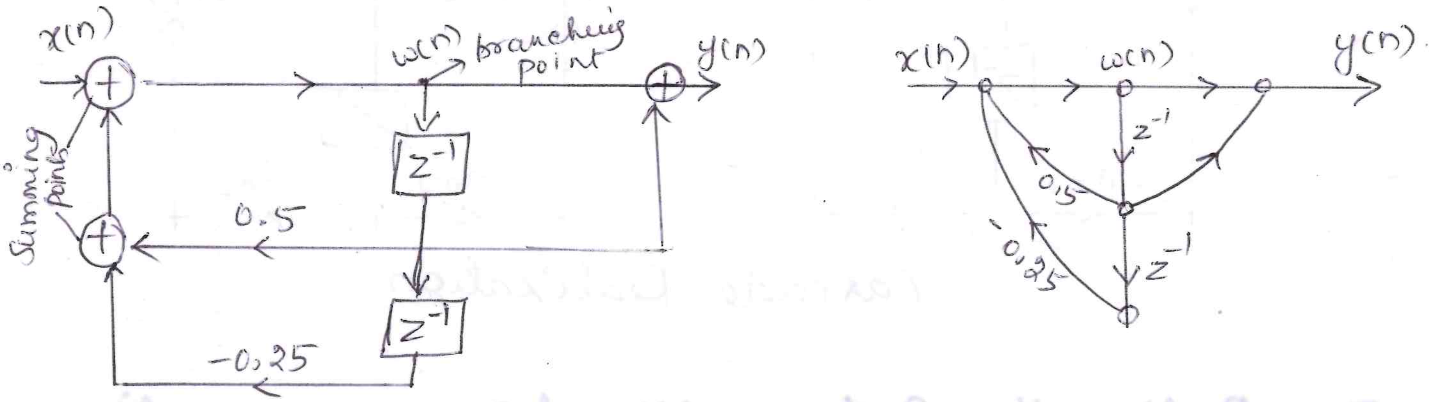
4) Transposition theorem and transposed Structure:

The transpose of a structure is defined by the following operations:

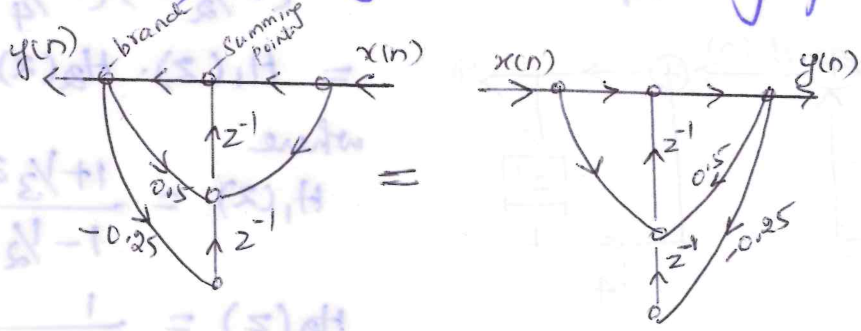
i) Reverse the direction of all branches in sig. flow graph
ii) Interchange the inputs & outputs
iii) Reverse the roles of all nodes in the flowgraph.
iv) Summing points become branching points
v) Branching points become summing points.

Ex: Determine the direct form II and Transposed direct form II for the system $y(n) = \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) + x(n) + x(n-1)$.

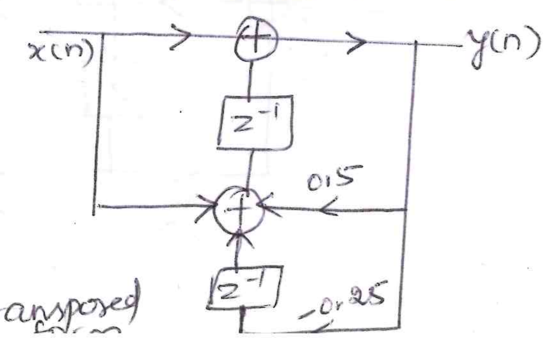
Soln: System function $H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}$



To get transposed direct form II do the following operations.
i) change the direction of all branches
ii) Interchange the input & output.
iii) change the summing point to branching point and vice versa.



Steps of operations in transposition



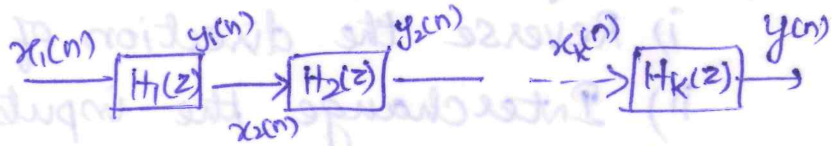
Transposed form

5) Cascade Form:

Consider IIR system with system function

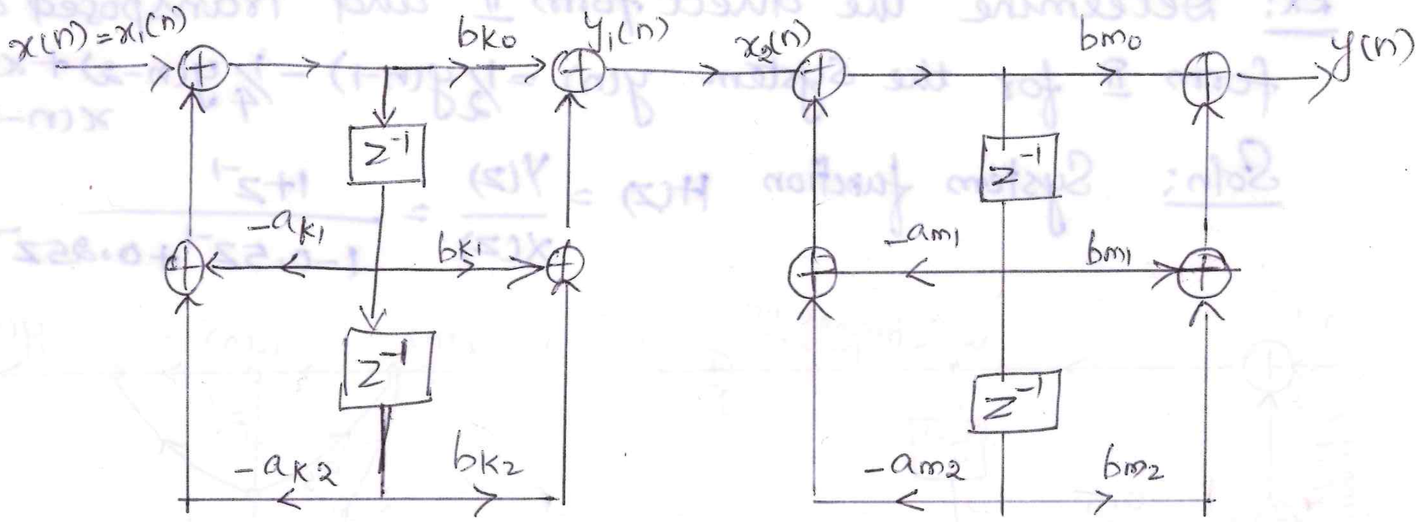
$$H(z) = H_1(z)\, H_2(z) \cdots H_K(z)$$

In block diagram,



$$H(z) = \frac{(b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2})(b_{m0} + b_{m1} z^{-1} + b_{m2} z^{-2})}{(1 + a_{k1} z^{-1} + a_{k2} z^{-2})(1 + a_{m1} z^{-1} + a_{m2} z^{-2})}$$

$$= H_1(z)\, H_2(z).$$

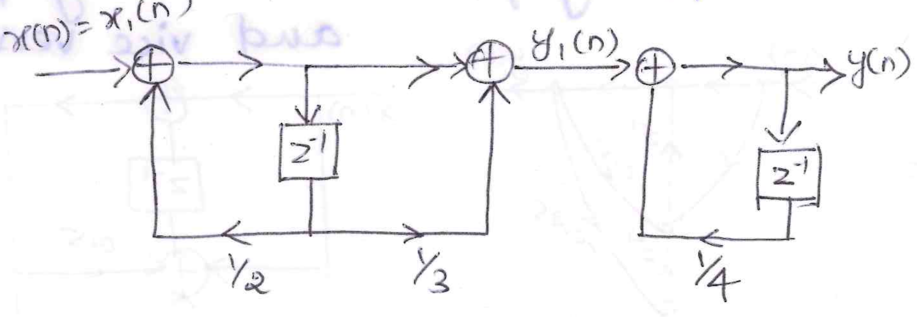


Cascade Realization.

Ex: Realize the system with difference equation

$$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{3} x(n-1)$$ in cascade form.

Soln: $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 + \frac{1}{3} z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}} = \dfrac{1 + \frac{1}{3} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{4} z^{-1})}$



$= H_1(z) \cdot H_2(z)$

where

$$H_1(z) = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$H_2(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}.$$

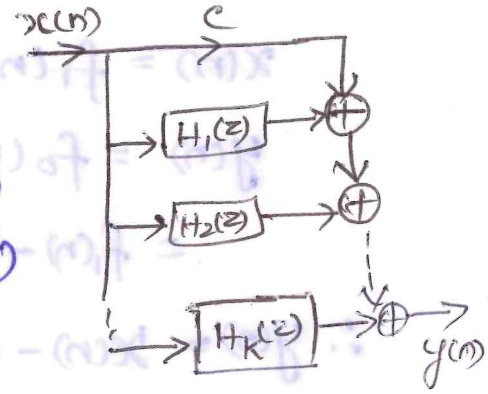
6) Parallel Form Structure:

A parallel form realization of an IIR system can be obtained by performing a partial expansion of $H(z) = c + \sum_{k=1}^{N} \frac{c_k}{1-P_k z^{-1}}$; where P_k - poles.

$$H(z) = c + \frac{c_1}{1-P_1 z^{-1}} + \frac{c_2}{1-P_2 z^{-1}} + \cdots + \frac{c_N}{1-P_N z^{-1}}$$

$$H(z) = \frac{Y(z)}{X(z)} = c + H_1(z) + H_2(z) + \cdots H_N(z)$$

$$\therefore Y(z) = cX(z) + H_1(z)X(z) + H_2(z)X(z) + \cdots + H_N(z)X(z)$$



Parallel Form

Ex: Realize the system given by difference equation $y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$ in parallel form.

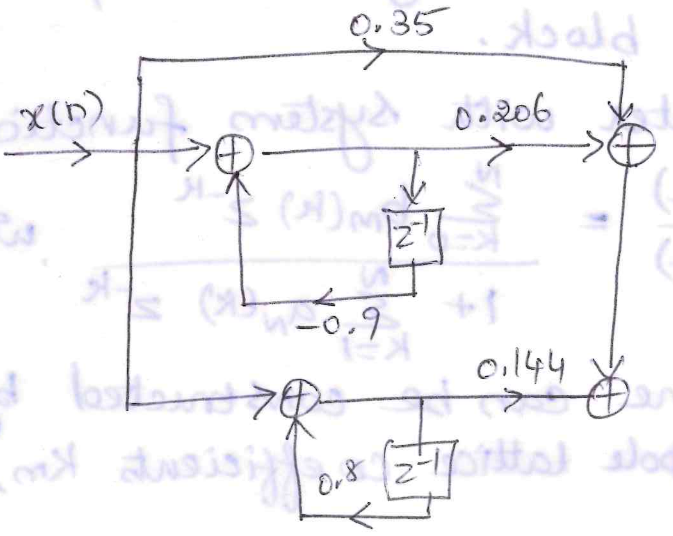
Soln: $$H(z) = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

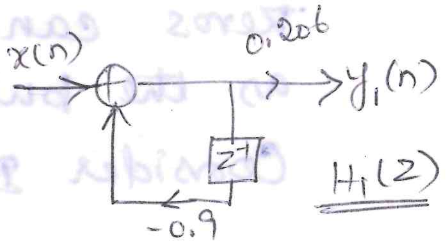
$-0.72z^{-2} + 0.1z^{-1} + 1 \;\big)\; -0.252z^{-2} + 0.7 \;\big(\; 0.35$
$-0.252z^{-2} + 0.035z^{-1} + 0.35$
$-0.035z^{-1} + 0.35$

$$= 0.35 + \frac{0.35 - 0.035z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

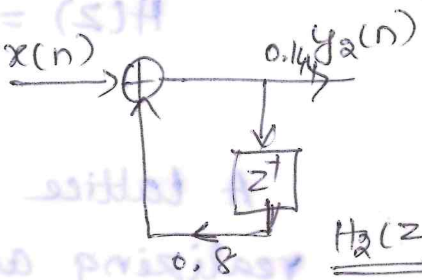
$$= 0.35 + \frac{0.206}{1+0.9z^{-1}} + \frac{0.144}{1-0.8z^{-1}}$$

$$= c + H_1(z) + H_2(z)$$





$H_1(z)$



$H_2(z)$

7) Lattice Structure of IIR System:

Consider all-pole system with system function

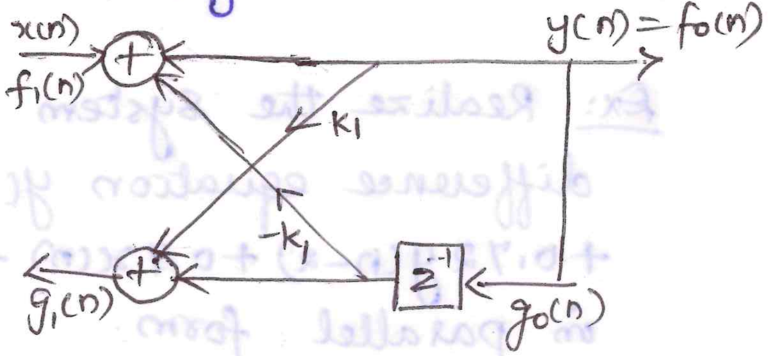
$$H(z) = \frac{1}{1+\sum_{k=1}^{N} a_N z^{-k}} = \frac{1}{A_N(z)}$$

The difference equation for this IIR system is

$$y(n) = -\sum_{k=1}^{N} a_N(k)\, y(n-k) + x(n)$$

$$\therefore x(n) = y(n) + \sum_{k=1}^{N} a_N(k)\, y(n-k)$$

For $N=1$, $x(n) = y(n) + a_1(1)\, y(n-1)$



$x(n) = f_1(n)$

$y(n) = f_0(n)$

$= f_1(n) - K_1 g_0(n-1)$

$\therefore y(n) = x(n) - K_1 y(n-1)$

$\Rightarrow x(n) = y(n) + K_1 y(n-1)$

$g_1(n) = K_1 f_0(n) + g_0(n-1) = K_1 y(n) + y(n-1)$ $[\because K_1 = a_1(1)]$

8) Lattice-Ladder Structure:

A general IIR filter containing both poles & zeros canbe realized using an all pole lattice as the building block.

Consider IIR filter with system function

$$H(z) = \frac{B_M(z)}{A_N(z)} = \frac{\sum_{k=0}^{N} b_M(k)\, z^{-k}}{1+\sum_{k=1}^{N} a_N(k)\, z^{-k}}, \quad \text{where } N \geq M$$

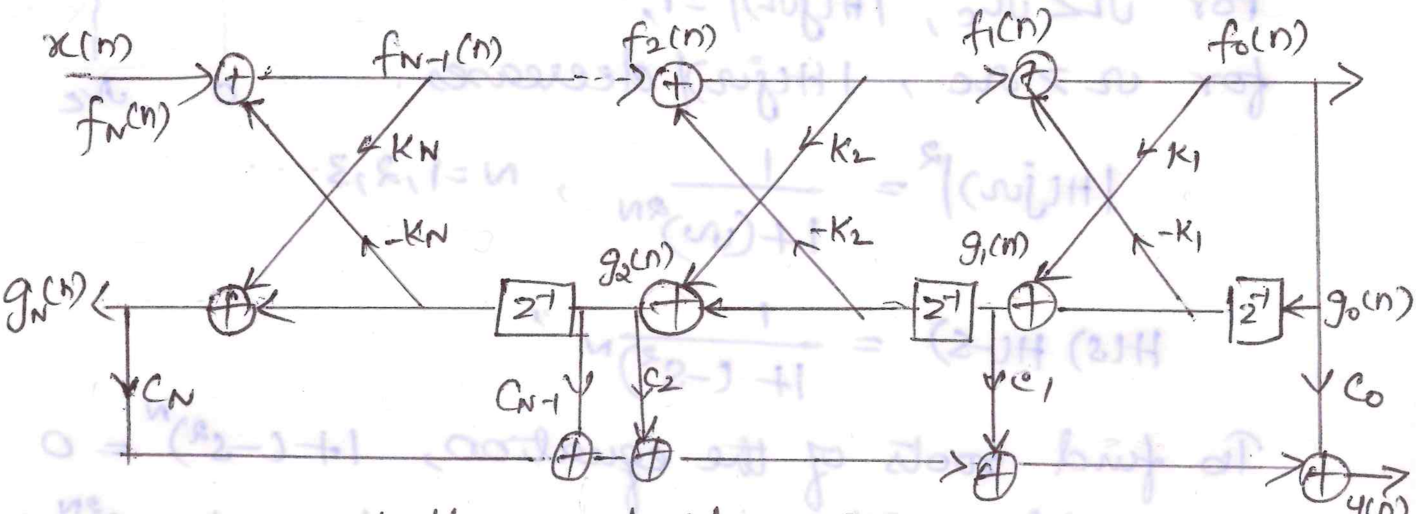
A lattice structure can be constructed by first realizing an all pole lattice coefficients K_m, $1 \leq m \leq N$

for the denominator $A_N(z)$, and then adding a ladder part for $M=N$. O/p of ladder part can be expressed as weighted combination of $\{g_m(n)\}$

O/p $y(n) = \sum_{m=0}^{N} c_m\, g_m(n)$.

where c_m - ladder coefficients

$$c_m = b_m - \sum_{i=m+1}^{M} c_i\, a_i(i-m); \quad m = M, M-1 \cdots 0$$



Lattice - Ladder Structure

Analog Lowpass Filter Design:

Analog filter transfer function is

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i}$$

Laplace transform of impulse response $h(t)$ is

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt.$$

Types of Analog filter design:

1. Butterworth filter
2. Chebyshev filter

Characteristics of commonly used analog filters

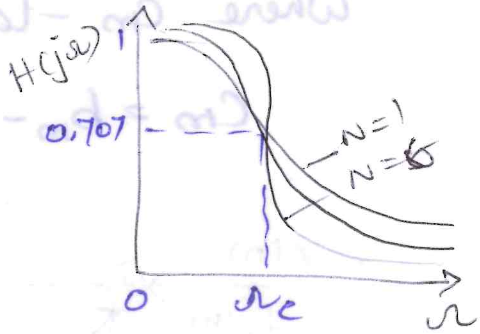
10 **Analog lowpass Butterworth filter:**

The magnitude function of LPF is

$$H(j\Omega) = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{1/2}}, \quad N = 1, 2, 3 \dots$$

N - order of filter
Ω_c - Cut off freq.

This function is monotonically decreasing, where max. response is unity at $\Omega = 0$



For $\Omega < \Omega_c$, $|H(j\Omega)| = 1$,
for $\Omega > \Omega_c$, $|H(j\Omega)|$ decreases.

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2N}}, \quad N = 1, 2, 3 \dots$$

$$H(s)\,H(-s) = \frac{1}{1+(-s^2)^N},$$

To find roots of the equation, $1+(-s^2)^N = 0$

For odd, $s^{2N} = 1$, then it reduces to $s^{2N} = 1 = e^{j2\pi k}$.

$$s_k = e^{j\pi k/N}, \quad k = 1, 2 \dots 2N$$

For N even, $s^{2N} = -1 = e^{j(2k-1)\pi}$ which gives

$$s_k = e^{j(2k-1)\pi/2N} \quad \text{for } k = 1, 2 \dots 2N.$$

Thus, for N odd,

$$s_1 = e^{j\pi/3} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = 0.5 + j0.866$$

$$s_2 = e^{j2\pi/3} = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -0.5 + j0.866$$

$$s_3 = e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$$s_4 = e^{j4\pi/3} = \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = -0.5 - j0.866$$

$$s_5 = e^{j5\pi/3} = \cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3} = 0.5 - j0.866$$

$$s_6 = e^{j2\pi} = \cos 2\pi + j\sin 2\pi = 1$$

$$\therefore \boxed{S_k = e^{j\phi_k}} \quad \text{where } \phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}, \; k = 1, 2 \cdots N$$

List of Butterworth polynomials:

N	Denominator of H(s)
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.84775s+1)$.

Ex: For $N=3$, $(s+1)\{(s+0.5)^2+(0.866)^2\} = (s+1)(s^2+s+1)$

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 (\Omega/\Omega_p)^{2N}}$$

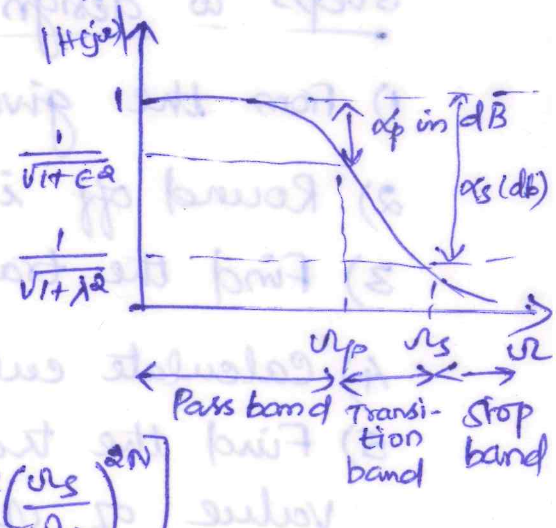
$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right]$$

$$20\log|H(j\Omega_p)| = -\alpha_p = -10\log(1+\epsilon^2)$$

$$\therefore \alpha_p = 10\log(1+\epsilon^2)$$

$$1+\epsilon^2 = 10^{0.1\alpha_p}$$

$$\epsilon = (10^{0.1\alpha_p}-1)^{1/2}$$



Stop band attenuation

$$20\log|H(j\Omega_s)| = 10\log 1 - 10\log\left[1+\epsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$-\alpha_s = -10\log\left[1+\epsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$0.1\alpha_s = \log\left[1+\epsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right]$$

$$\epsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s} - 1$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$$

$\therefore$ Order of the filter $N = \dfrac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}}$

Round off N to next integer.

$$\therefore \; N \geq \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} \geq \frac{\log(\lambda/\epsilon)}{\log\dfrac{\Omega_s}{\Omega_p}}$$

where $\epsilon = (10^{0.1\alpha_p}-1)^{1/2}$, $\lambda = (10^{0.1\alpha_s}-1)^{1/2}$.

$$A = \frac{\lambda}{\epsilon} = \left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)^{1/2} \quad \& \quad K = \frac{\Omega_p}{\Omega_s}$$

$\therefore$ Order of lowpass butterworth filter $N \geq \dfrac{\log A}{\log(1/k)}$.

Steps to design an analog Butterworth LPF:

1) From the given specifications find the order of filter N.
2) Round off it to the next nearest integer.
3) Find the transfer function H(s) for $\Omega_c = 1$ rad/sec for the values of N.
4) Calculate cutoff frequency Ω_c.
5) Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting $s \to \dfrac{s}{\Omega_c}$ in H(s).

Ex.1): Design an analog butterworth filter that has a -2 db passband attenuation at a frequency of 20 rad/sec and atleast -10 db stop band attenuation at 30 rad/sec.

Soln: $\alpha_p = 2$ db ; $\Omega_p = 20$ rad/sec

$\alpha_s = 10$ db ; $\Omega_s = 30$ rad/sec.

$$N \geq \log \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}} \Big/ \log \frac{\Omega_s}{\Omega_p}$$

$$\geq \frac{\log \sqrt{\frac{10-1}{10^{0.2}-1}}}{\log \frac{30}{20}} \geq 3.37$$

Round off N, $\therefore N = 4$.

Normalised HP butterworth filter for $N=4$ as

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.84776s+1)}$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{20}{(10^{0.2}-1)^{1/8}} = 21.3868$$

The transfer function for $\Omega_c = 21.3868$ can be obtained by substituting $s \rightarrow \frac{s}{21.3868}$ in $H(s)$

(ie) $$H(s) = \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right)+1} \times \frac{1}{\left(\frac{s}{21.3868}\right)^2 + \left(\frac{s}{21.3868}\right)1.8477+1}$$

$$= \frac{0.20921 \times 10^6}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

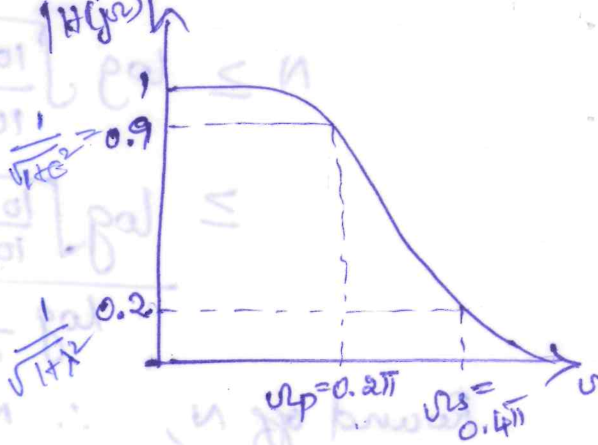
Ex.2: For the given specifications design an analog butterworth filter $0.9 \leq |H(j\Omega)| \leq 1$ for $0 \leq \Omega \leq 0.2\pi$. $|H(j\Omega)| \leq 0.2$ for $0.4\pi \leq \Omega \leq \pi$.

Soln: $\Omega_p = 0.2\pi$, $\Omega_s = 0.4\pi$, $\frac{1}{\sqrt{1+\epsilon^2}} = 0.9$, $\frac{1}{\sqrt{1+\lambda^2}} = 0.2$

from this, $\epsilon = 0.484$, $\lambda = 4.898$

$$N \geq \frac{\log\left(\frac{1}{\epsilon}\right)}{\log \frac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log 4.898/0.484}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.34$$



(ie) $N = 4$

Transfer function $H(s) = \dfrac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{\Omega_p}{\epsilon^{1/N}} = \frac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi$$

$H(s)$ for $\Omega_c = 0.24\pi$ can be obtained by substituting $s \rightarrow \dfrac{s}{0.24\pi}$ in $H(s)$.

(ie)
$$H(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 0.76537\left(\frac{s}{0.24\pi}\right)+1\right\}} \times \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 1.8477\left(\frac{s}{0.24\pi}\right)+1\right\}}$$

$$= \frac{0.323}{(s^2+0.577s+0.0576\pi^2)(s^2+1.393s+0.0576\pi^2)}$$

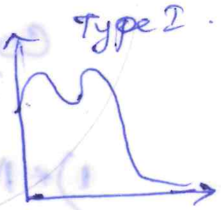
Analog Lowpass Chebyshev Filters:

Type I : All-pole filters that exhibit equiripple behaviour in the pass band and a monotonic characteristics in the stop band.

Type II : - contains both poles & zeros and exhibits a monotonic behaviour in the pass band and an equiripple behaviour in the stopband.

Magnitude of N^{th} order type I filter is

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \qquad N = 1, 2 \cdots$$



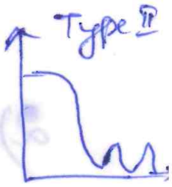
N^{th} order chebyshev polynomial

$C_N(x) = \cos(N\cos^{-1}x)$, $|x| \le 1$ (Pass band)

$C_N(x) = \cosh(N\cosh^{-1}x)$, $|x| > 1$ (Stop band)

By recursive formula, $C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x)$, $N > 1$

where $C_0(x) = 1$ & $C_1(x) = x$.



Order $N \ge \dfrac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} \ge \dfrac{\cosh^{-1}A}{\cosh^{-1}(1/k)}$

Poles of chebyshev filter $s_k = a\cos\phi_k + jb\sin\phi_k$

$= \sigma_k + j\Omega_k$

where $a = \Omega_p\left[\dfrac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$, $b = \Omega_p\left[\dfrac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$

$\phi_k = \dfrac{\pi}{2} + \dfrac{(2k-1)\pi}{2N}$, $k = 1, 2 \ldots N$.

Type II filter:

$|H(j\Omega)|^2 = \dfrac{1}{1 + \epsilon^2\left[\dfrac{C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}{C_N^2\left(\frac{\Omega_s}{\Omega}\right)}\right]}$; $C_N(x)$ – N^{th} order polynomial

Ω_s – Stopband freq

Ω_p – Passband freq.

Zeros are on imaginary axis, $S_k = j\dfrac{\Omega_s}{\sin\phi_k}$, $k = 1$ to N

Poles are at (x_k, y_k), $x_k = \dfrac{\Omega_s\sigma_k}{\sigma_k^2 + \Omega_k^2}$, $y_k = \dfrac{\Omega_s\Omega_k}{\sigma_k^2 + \Omega_k^2}$.

where $\sigma_k = a\cos\phi_k$, $\Omega_k = b\sin\phi_k$, $\mu = \lambda + \sqrt{1+\lambda^2}$

Order of filter $N = \dfrac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \dfrac{\cosh^{-1}A}{\cosh^{-1}(1/k)}$

where $A = \lambda/\epsilon$, $k = \dfrac{\Omega_p}{\Omega_s}$

$\epsilon = (10^{0.1\alpha_p} - 1)^{1/2}$, $\lambda = (10^{0.1\alpha_s} - 1)^{1/2}$.

16 Comparison between Butterworth filter & Chebyshev filter:

1) * Magnitude response of Butterworth filter decreases monotonically as frequency Ω increases 0 to ∞.
* Magnitude response of chebyshev filter exhibits ripples in passband / stop band according to type.

2) The transition band is more in Butterworth than Chebyshev.

3) Poles of Butterworth lie on circle.
Poles of chebyshev lie on ellipse.

4) No. of poles in butterworth is more than chebyshev
(ie) Order of chebyshev is less than Butterworth.

Steps to design Analog chebyshev LPF:

1) From the specifications, find the order of filter N.
2) Round off it to nearest integer.
3) Find a, b (major + minor axis) of ellipse,

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2}, \quad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

where $\mu = \epsilon^{-1} + \sqrt{\epsilon^{-2}+1}$, $\epsilon = \sqrt{10^{0.1\alpha_p} - 1}$.

4) Calculate the poles which lie on ellipse.

$$S_K = a\cos\phi_K + jb\sin\phi_K, \quad k = 1, 2 \cdots N$$

Where $\phi_K = \pi/2 + \left(\frac{2k-1}{2N}\right)\pi$, $k = 1, 2 \cdots N$.

5) Find the denominator polynomial of transfer function using the above poles.
6) The numerator of the transfer function depends on N.

a) For N odd, substitute $s=0$ in the denominator and find value. The value is equal to numerator of transfer function.

b) For N even, substitute $s=0$ in the denominator and divide the result by $\sqrt{1+\epsilon^2}$. The value is equal to numerator.

Ex.1: Given $\alpha_p = 3\,db$, $\alpha_s = 16\,db$, $f_p = 1\,kHz$ and $f_s = 2\,kHz$. Determine the order of the filter using Chebyshev approximation. Find $H(s)$.

Soln: $\Omega_p = 2\pi \times 1000\,Hz = 2000\pi$ rad/sec.

$\Omega_s = 2\pi \times 2000\,Hz = 4000\pi$ rad/sec.

and $\alpha_p = 3\,db$, $\alpha_s = 16\,db$.

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1}\sqrt{\frac{10^{1.6}-1}{10^{0.3}-1}}}{\cosh^{-1}\frac{4000\pi}{2000\pi}} = 1.91$$

After round off, $N = 2$.

For N even, oscillatory curve starts from $\frac{1}{\sqrt{1+\epsilon^2}}$

$$\epsilon = (10^{0.1\alpha_p}-1)^{1/2} = (10^{0.3}-1)^{1/2} = 1.$$

$$\mu = \epsilon^{-1} + \sqrt{1+\epsilon^{-2}} = 2.414$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 2000\pi\left[\frac{(2.414)^{1/2} - (2.414)^{-1/2}}{2}\right] = 910\pi$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 2000\pi\left[\frac{(2.414)^{1/2} + (2.414)^{-1/2}}{2}\right] = 2197\pi$$

The poles are given by $s_k = a\cos\phi_k + jb\sin\phi_k$ $\quad k = 1,2$.

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}, \quad k = 1,2$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 135^\circ, \quad \phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = 225^\circ$$

$S_1 = a\cos\phi_1 + jb\sin\phi_1 = -643.46\pi + j1554\pi$

$S_2 = a\cos\phi_2 + jb\sin\phi_2 = -643.46\pi - j1554\pi$

The denominator of $H(s) = (s+643.46\pi)^2 + (1554\pi)^2$

The numerator of $H(s) = \dfrac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\epsilon^2}}$

$= (1414.38)^2\pi^2$

The transfer function $H(s) = \dfrac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$

Ex.2: Obtain an analog chebyshev filter transfer function that satisfies the constraints

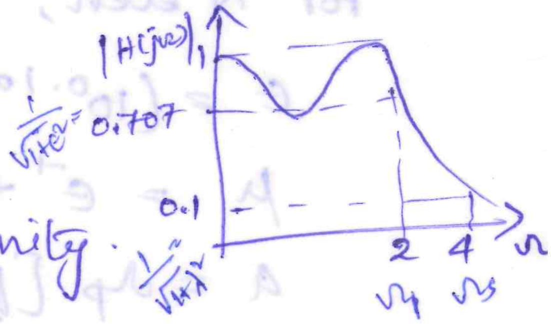
$\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1 \; ; \quad 0 \le \Omega \le 2$

$|H(j\Omega)| < 0.1 \; ; \quad \Omega \ge 4.$

Soln: $\dfrac{1}{\sqrt{1+\epsilon^2}} = \dfrac{1}{\sqrt{2}}$, $\dfrac{1}{\sqrt{1+\lambda^2}} = 0.1$, $\Omega_p = 2$, $\Omega_s = 4$.

$\therefore \epsilon = 1$, $\lambda = 9.95$

$N \ge \dfrac{\cosh^{-1}\lambda/\epsilon}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} = \dfrac{\cosh^{-1} 9.95}{\cosh^{-1} 2} = 2.269$



Round off N, $N = 3$.

For N odd, curve starts from unity.

$\mu = \epsilon^{-1} + \sqrt{1+\epsilon^{-2}} = 2.414$.

$a = \Omega_p\left[\dfrac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 2\left[\dfrac{(2.414)^{1/3} - (2.414)^{-1/3}}{2}\right] = 0.596$

$b = \Omega_p\left[\dfrac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 2\left[\dfrac{(2.414)^{1/3} + (2.414)^{-1/3}}{2}\right] = 2.087$

Poles of filter $\phi_k = \dfrac{\pi}{2} + \dfrac{(2k-1)\pi}{2N}$, $k = 1,2,3$

$\phi_1 = 120°$, $\phi_2 = 180°$, $\phi_3 = 240°$.

$$s_k = a\cos\phi_k + jb\sin\phi_k \quad , \quad k = 1, 2, 3$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1 = -0.298 + j1.807$$

$$s_2 = a\cos\phi_2 + jb\sin\phi_2 = -0.596$$

$$s_3 = a\cos\phi_3 + jb\sin\phi_3 = -0.298 - j1.807$$

The denominator polynomial is

$$(s+0.596)\{(s+0.298)-j1.807\}\{(s+0.298)+j1.807\}$$

$$= (s+0.596)\left[(s+0.298)^2 + (1.807)^2\right]$$

$$= (s+0.596)(s^2+0.596\,s+3.354)$$

The numerator is obtained by substitute $s=0$ in the denominator. (for N odd)

$\therefore$ Numerator of $H(s) = 2$.

Transfer function $H(s) = \dfrac{2}{(s+0.596)(s^2+0.596s+3.354)}$

Freq. Transformation in Analog domain:

LPF to LPF : $s \rightarrow \dfrac{s}{\Omega_c}$

LPF to HPF : $s \rightarrow \dfrac{\Omega_c}{s}$

LPF to BPF : $s \rightarrow \dfrac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}$

$$\frac{\Omega_s}{\Omega_p} = \Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{-\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)} \; ; \quad B = \frac{\Omega_2^2 - \Omega_l\Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$

LPF to BSF : $s \rightarrow \dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l\Omega_u}$

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u} \; ; \quad B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l\Omega_u}$$

Design of IIR Filters from Analog filters:

1) Approximation of Derivatives:

For digitizing an analog filter into digital filter is to approximate the differential equation by an equivalent difference equation.

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T}$$

T - sampling interval and $y(n) \equiv y(nT)$

Laplace transform of $\frac{dy(t)}{dt} = s\,Y(s)$

$$y(t) \longrightarrow \boxed{H(s)=s} \longrightarrow \frac{dy(t)}{dt}$$

Z-transform of $\frac{y(n) - y(n-1)}{T}$ is $\frac{(1-z^{-1})\,Y(z)}{T}$

$$\therefore\ s = \frac{1-z^{-1}}{T}$$

$$y(t) \longrightarrow \boxed{\frac{1-z^{-1}}{T}} \longrightarrow \frac{y(n)-y(n-1)}{T}$$

$\therefore$ System function $H(z) = \left.H(s)\right|_{s=\frac{1-z^{-1}}{T}}$

$$\therefore\ z = \frac{1}{1-sT} = \frac{1}{1-j\Omega T} = \frac{1+j\Omega T}{1+\Omega^2 T^2}$$

$$= \frac{1}{1+\Omega^2T^2} + j\,\frac{\Omega T}{1+\Omega^2 T^2} = x + jy$$

$x^2 + y^2 = x$ which canbe written as

$$(x - \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}.$$

Characteristics:
1) Left half of s-plane maps inside a circle of radius ½ centred at z = ½ in z-plane.
2) Right half maps to outside circle
3) The $j\Omega$ axis maps on perimeter of circle.

2) Impulse Invariance method:

In this method, IIR filter is designed that unit impulse response $h(n)$ of digital filter is the sampled version of impulse response of analog filter.

Z-transform of infinite impulse response is

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} h(n) e^{-sTn}.$$

(ie) $z = e^{sT}$, $s = \sigma + j\Omega$, $z = re^{j\omega}$

$$re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T}$$

from this, $r = e^{\sigma T}$, $\omega = \Omega T$

The real part of analog poles determine the radius of z-plane pole & Imaginary part of analog pole indicates angle of digital pole.

Let $H_a(s)$ - System function of an analog filter.

$$H_a(s) = \sum_{K=1}^{N} \frac{C_K}{s - P_K} \; ;$$

C_K - coefficient in partial fraction expansion

P_K - poles of analog filter

$$\therefore h_a(t) = \sum_{K=1}^{N} C_K e^{P_K t} \quad , t \geq 0$$

At $t = nT$, $h_a(nT) = \sum_{K=1}^{N} C_K e^{P_K nT}$.

But, $H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$.

$$= \sum_{n=0}^{\infty} \sum_{K=1}^{N} C_K e^{P_K nT} z^{-n}.$$

$$= \sum_{K=1}^{N} C_K \sum_{n=0}^{\infty} (e^{P_K T} z^{-1})^n = \sum_{K=1}^{n} \frac{C_K}{1 - e^{P_K T} z^{-1}}$$

(ie) $H_a(S) = \sum_{K=1}^{N} \frac{C_K}{S-P_K}$ then $H(z) = \sum_{K=1}^{N} \frac{C_K}{1-e^{P_K T} z^{-1}}$.

For Sampling rate, $H(z) = \sum_{K=1}^{N} \frac{T C_K}{1-e^{P_K T} z^{-1}}$.

Steps: 1) For the given specifications, find $H_a(S)$, the transfer function of an analog filter.

2) Select the sampling rate of digital filter, T sec. per sample.

3) Express the analog filter transfer function as the sum of single pole filters.

$$H_a(S) = \sum_{K=1}^{N} \frac{C_K}{S-P_K}.$$

4) Compute the z-transform of the digital filter

$$H(z) = \sum_{K=1}^{N} \frac{C_K}{1-e^{P_K T} z^{-1}}.$$

For high Sampling rate, $H(z) = \sum_{K=1}^{N} \frac{T C_K}{1-e^{P_K T} z^{-1}}$.

Ex: For the analog transfer function $H(S) = \frac{2}{(S+1)(S+2)}$ determine $H(z)$ using impulse invariant method. Assume $T = 1$ sec.

$$H(S) = \frac{2}{(S+1)(S+2)} = \frac{A}{S+1} + \frac{B}{S+2} = \frac{2}{S+1} - \frac{2}{S+2}$$

Using impulse invariance, $H(S) = \sum_{K=1}^{N} \frac{C_K}{S-P_K}$;

$P_1 = -1$, $P_2 = -2$.

$$\therefore H(z) = \frac{2}{1-e^{-T} z^{-1}} - \frac{2}{1-e^{-2T} z^{-1}}$$

For $T = 1$ sec, $H(z) = \frac{2}{1-e^{-1} z^{-1}} - \frac{2}{1-e^{-2} z^{-1}} = \frac{0.465 z^{-1}}{1-0.503 z^{-1} + 0.049 z^{-2}}$

3) Bilinear Transformation:

This method is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the z-plane.

Consider analog filter with system function

$$H(s) = \frac{b}{s+a}$$

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$s\,Y(s) + a\,Y(s) = bX(s).$$

(ie) $$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t).$$

$y(t)$ can be approximated by trapezoidal formula

$$y(t) = \int_{t_0}^{t} y'(\tau)\,d\tau + y(t_0)$$

At $t = nT$, $t_0 = nT-T$, $$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T)$$

$$y'(nT) = -a\,y(nT) + bx(nT)$$

$$\therefore\ y(nT) = \frac{T}{2}\left[-ay(nT) + bx(nT) - ay(nT-T) + bx(nT-T)\right] + y(nT-T)$$

$$y(nT) + \frac{aT}{2}y(nT) - \left(1 - \frac{aT}{2}\right)y(nT-T) = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

With $y(n) = y(nT)$ and $x(n) = x(nT)$, we obtain the result

$$\left(1+\frac{aT}{2}\right)y(n) - \left(1-\frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

The z-transform of this equation is

$$\left(1+\frac{aT}{2}\right)Y(z) - \left(1-\frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}\left(1+z^{-1}\right)X(z).$$

System function of digital filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}(1+z^{-1})}{1+\frac{aT}{2} - \left(1-\frac{aT}{2}\right)z^{-1}}$$

$$= \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})}$$

Dividing numerator & denominator by $\frac{T}{2}(1+z^{-1})$, we get

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

Mapping s-plane to z-plane, $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

The relationship between s & z is known as bilinear transformation.

Let $z = re^{j\omega}$, $s = \sigma + j\Omega$

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + j2r\sin\omega + r^2\sin^2\omega}{1+r^2\cos^2\omega + 2r\cos\omega + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

$$\therefore \sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right]; \quad \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}} = \frac{2}{T}\tan\frac{\omega}{2}.$$

Let ω & Ω are frequency in digital & analog filter,

$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$, For small values, $\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T}$

$\boxed{\omega = \Omega T}$, For low frequencies, Ω & ω are linear.

For high frequencies, Ω & ω are non linear is distortion is introduced in frequency scale of digital filter to analog filter. This is known as Warping effect.

This can be eliminated by prewarping the analog filter.

Steps : 1) From the given specifications, find prewarping analog frequencies using $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$.

2) Using analog frequencies, find $H(s)$ of analog filter.

3) Select the sampling rate of the digital filter, call it T sec. per sample.

4) Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ into transfer function found in step 2.

Ex: Apply bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with $T = 1$ sec and find $H(z)$.

Soln: $$H(s) = \frac{2}{(s+1)(s+2)}$$

Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$ to get $H(z)$.

$$H(z) = \frac{2}{\left\{\frac{2(1-z^{-1})}{1+z^{-1}}+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

(At $T = 1$ sec)

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})4}$$

$$= \frac{(1+z^{-1})^2}{(6-2z^{-1})} = \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}.$$

4) The matched z-transform:

This method for converting analog into digital filter is to map the poles and zeros of $H(s)$ into poles and zeros in the z-plane.

If $$H(s) = \frac{\prod_{k=1}^{M}(s-z_k)}{\prod_{k=1}^{N}(s-p_k)}$$ where $\{z_k\}$ - zeros, $\{p_k\}$ - Poles.

$$\therefore H(z) = \frac{\prod_{k=1}^{M}(1-e^{z_k T}z^{-1})}{\prod_{k=1}^{N}(1-e^{p_k T}z^{-1})},$$ T - Sampling interval

Thus each factor of the form $(s-a)$ in $H(s)$ is mapped into the factor $1-e^{aT}z^{-1}$. This mapping is called matched transform.

<u>Frequency Transformation in Digital Domain</u>:

A digital LPF can be converted into digital HPF, BPF, & BSF.

1) LPF to HPF: $z^{-1} \to \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$ where $\alpha = \frac{\sin[(\omega_p-\omega_p')/2]}{\sin[(\omega_p+\omega_p')/2]}$

ω_p - Passband freq of LPF.

ω_p' - Passband frequency of new LPF.

2) LPF to HPF: $z^{-1} = -\left[\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right]$ where $\alpha = -\frac{\cos[(\omega_p'+\omega_p)/2]}{\cos[(\omega_p'-\omega_p)/2]}$

ω_p - Pass band of LPF

ω_p' - Pass band freq. of HPF

3) LPF to BPF :

$$z^{-1} \rightarrow \frac{-\left(z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + \frac{k-1}{k+1}\right)}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$$

where $\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$

$$k = \cot\left[\frac{\omega_u - \omega_l}{2}\right] \tan\frac{\omega_p}{2}$$

ω_u – Upper cutoff frequency, ω_l – Lower cutoff frequency.

4) LPF to BSF :

$$z^{-1} \rightarrow \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$$

where $\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$,

$$k = \tan[(\omega_u - \omega_l)/2] \tan\frac{\omega_p}{2}.$$

<u>Ex:</u> Convert the single pole LPF with system function $H(z) = \frac{0.5(1+z^{-1})}{1-0.302z^{-1}}$, into BPF with upper & lower cutoff frequencies ω_u & ω_l respectively. The LPF has 3db bandwidth $\omega_p = \frac{\pi}{6}$ and $\omega_u = \frac{3\pi}{4}$, $\omega_l = \pi/4$

<u>Soln:</u> Digital to digital transformation from LPF to BPF is

$$z^{-1} \rightarrow \frac{-\left(z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + \frac{k-1}{k+1}\right)}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$$

$$K = \cot\left[\frac{\omega_u - \omega_l}{2}\right]\tan\frac{\omega_P}{2} = \cot\left(\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{2}\right)\tan\frac{\pi}{6\times 2}$$

$$= \cot(\pi/4)\tan\frac{\pi}{12} = 0.268$$

$$\alpha = \frac{\cos\left(\frac{\omega_u + \omega_l}{2}\right)}{\cos\left(\frac{\omega_u - \omega_l}{2}\right)} = \frac{\cos\left(\frac{\frac{3\pi}{4} + \frac{\pi}{4}}{2}\right)}{\cos\left(\frac{\frac{3\pi}{4} - \pi/4}{2}\right)} = \frac{\cos\frac{\pi}{2}}{\cos\pi/4} = 0$$

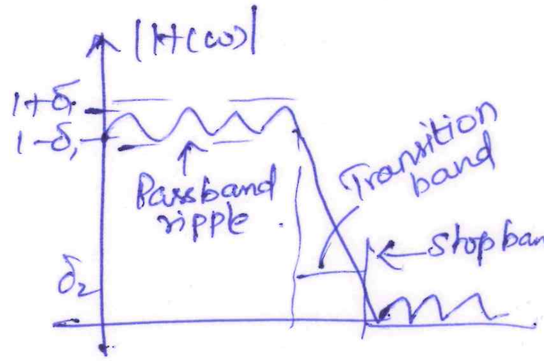
$$\therefore z^{-1} = \frac{-\left(z^{-2} + \frac{0.268-1}{0.268+1}\right)}{\frac{0.268-1}{0.268+1}z^{-2} + 1} = \frac{-z^{-2} + 0.577}{-0.577z^{-2} + 1}$$

Now, $$H(z) = \frac{0.5\left[1 + \frac{-z^{-2} + 0.577}{1 - 0.577z^{-2}}\right]}{1 - 0.302\left[\frac{-z^{-2} + 0.577}{1 - 0.577z^{-2}}\right]}$$

$$H(z) = \frac{0.955(1 - z^{-2})}{(1 - 0.333z^{-2})}$$

Characteristics of Practical Frequency-selective Filters:

Ideal filters are non causal & unrealizable for real time signal processing applications. $H(\omega)$ can't have an infinitely sharp cutoff from pass band to stop band. $|H(\omega)|$ is const. in passband of ideal filter. It is not necessary for filter response $|H(\omega)|$ to be zero in stopband, small amount of ripple is tolerable.



Transition band → Transition of freq. response from passband to SB.

Bandwidth → width of transition band $\omega_s - \omega_p$.

Band edge freq. ω_p, Stop band freq. ω_s

Ripple in pass band δ_1, in SB → δ_2.

In any filter design, Specify

1) max. tolerable passband ripple
2) passband edge freq. ω_p
3) max. tolerable stopband ripple
4) Stop band edge freq. ω_s

UNIT III. FIR FILTER DESIGN.

INTRODUCTION:

1) FIR filters are always stable.
2) FIR filters with linear phase can be easily designed.
3) FIR filters can be realized in both recursive and non-recursive structures.
4) FIR filters are free of limit cycle oscillations, when implemented on a finite wordlength digital system.
5) Excellent design methods are available.

Disadvantages: 1) Memory requirement & execution time are very high.
2) The implementation of narrow transition band FIR filter are very costly.

Structures of FIR:

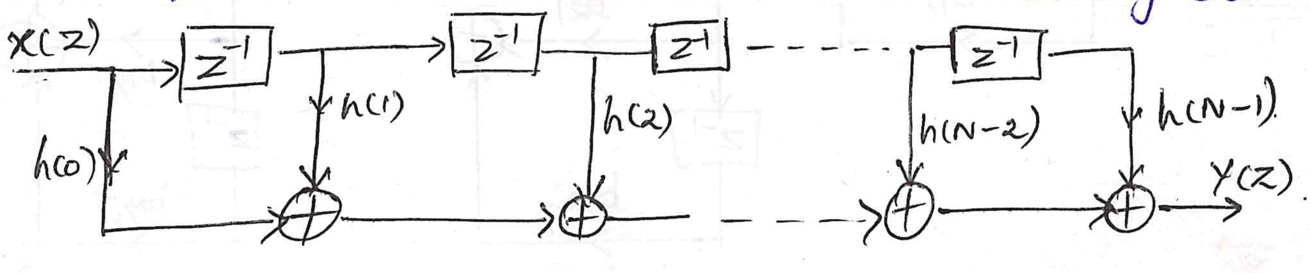
1) Transversal Structure (Direct form): The system function of FIR filter can be written as

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}.$$

$$= h(0) + h(1) z^{-1} + h(2) z^{-2} + \cdots + h(N-1) z^{-(N-1)}.$$

$$\therefore\ Y(z) = h(0) X(z) + h(1) z^{-1} X(z) + \cdots + h(N-1) z^{-(N-1)} X(z).$$

This structure is transversal (or) direct form.
This requires N multipliers, N-1 adders & (N-1) delay element

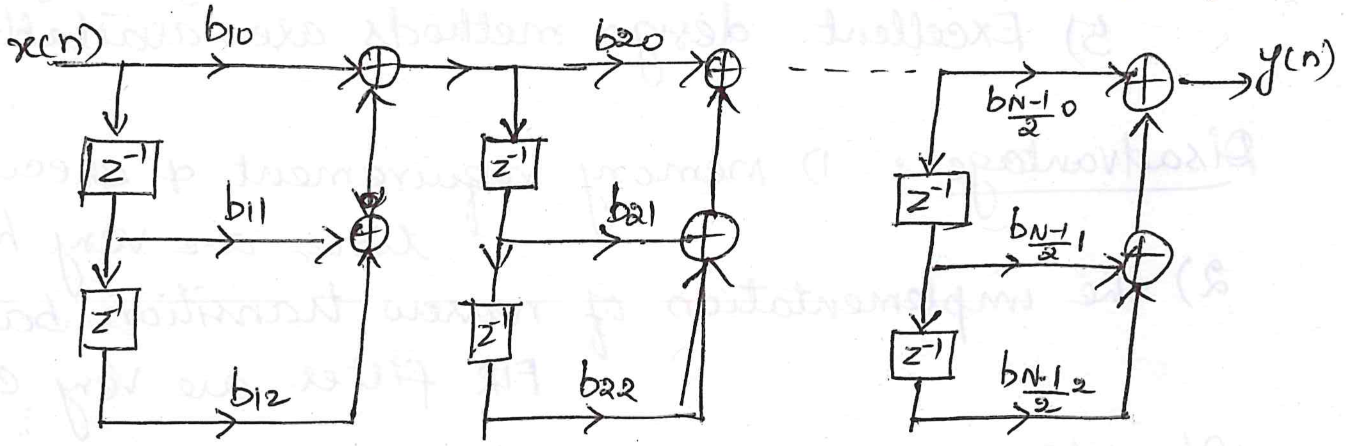


2) Cascade realization:

For N odd, $H(z) = \prod_{k=1}^{\frac{N-1}{2}} (b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2})$

$$= (b_{10} + b_{11} z^{-1} + b_{12} z^{-2})(b_{20} + b_{21} z^{-1} + b_{22} z^{-2}) \cdots$$
$$\times \left(b_{(\frac{N-1}{2})0} + b_{(\frac{N-1}{2})1} z^{-1} + b_{(\frac{N-1}{2})2} z^{-2} \right)$$

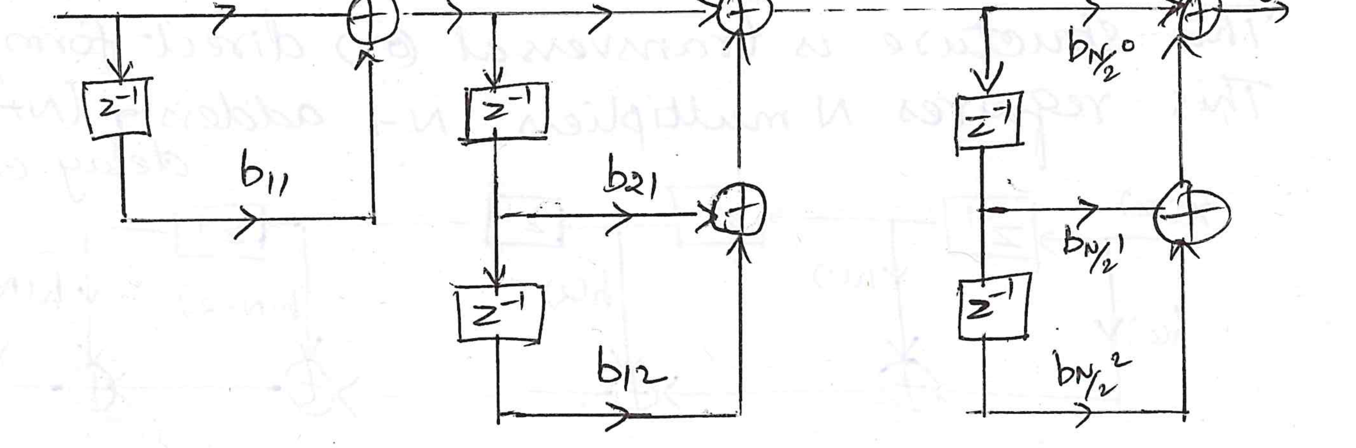
For N odd, N-1 will be even and H(z) will have $(N-1)/2$ second order factors. Each 2nd order factored form of H(z) is realized in direct form & is cascaded to realize H(z).



For N even, $H(z) = (b_{10} + b_{11} z^{-1}) \prod_{k=2}^{N/2} (b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2})$

When N even, N-1 is odd & H(z) has 1st order factor & $\left(\frac{N-2}{2}\right)$ second order factors.

$$H(z) = (b_{10} + b_{11} z^{-1})(b_{20} + b_{21} z^{-1} + b_{22} z^{-2})(b_{30} + b_{31} z^{-1} + b_{32} z^{-2}) \cdots$$
$$\times (b_{\frac{N}{2}0} + b_{\frac{N}{2}1} z^{-1} + b_{\frac{N}{2}2} z^{-2}).$$



Design of FIR Filters:
Symmetric and Antisymmetric filters:

An FIR filter of length M with i/p $x(n)$ & o/p $y(n)$ is described by difference equ

$$y(n) = b_0 x(n) + b_1 x(n-1) + \dots + b_{M-1} x(n-M+1)$$

$$= \sum_{k=0}^{M-1} b_k x(n-k) \qquad b_k \text{ - set of filter coeff.}$$

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k) \qquad h(n) \text{ - unit impulse respns}$$

$$b_k = h(k).$$

System fn $H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}$.

An FIR filter has linear phase if its unit sample response satisfies the condition

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1 \dots M-1.$$

$$H(z) = z^{-(M-1)/2} \left\{ h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{(M-3)/2} h(n) \left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2} \right] \right\} \rightarrow M\text{odd}$$

$$= z^{-(M-1)/2} \sum_{n=0}^{(M/2)-1} h(n) \left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2} \right] \rightarrow M\text{eve}$$

$$z^{-(M-1)} H(z^{-1}) = \pm H(z).$$

Roots of polynomial $H(z)$ are identical to the roots of polynomial $H(z^{-1})$.

When $h(n) = h(M-1-n)$, $H(\omega) = H_r(\omega) e^{-j\omega(M-1)/2}$.

$H_r(\omega)$ is a real fn of ω.

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{(M-3)/2} h(n) \cos\omega\left(\frac{M-1}{2} - n\right), \quad M\text{-odd}$$

$$= 2 \sum_{n=0}^{(M/2)-1} h(n) \cos\omega\left(\frac{M-1}{2} - n\right), \quad M\text{-even}.$$

Phase $\theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & , \text{ if } H_r(\omega) > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & \text{ if } H_r(\omega) < 0 \end{cases}$

when $h(n) = -h(M-1-n)$, the unit impulse response is <u>antisymmetric</u>.

For M odd, $h\left(\frac{M-1}{2}\right) = 0$.

For M even, each term $h(n)$ has a matching term of opposite sign.

$$H(\omega) = H_r(\omega)\, e^{j[-\omega(M-1)/2 + \pi/2]}$$

where $H_r(\omega) = 2 \sum_{n=0}^{(M-3)/2} h(n) \sin \omega\left(\frac{M-1}{2} - n\right)$ M-odd.

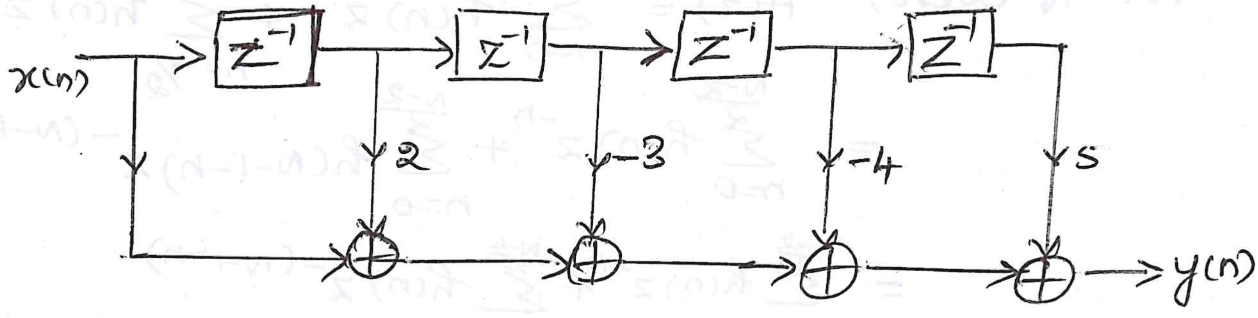
$$H_r(\omega) = 2 \sum_{n=0}^{(M/2)-1} h(n) \sin \omega\left(\frac{M-1}{2} - n\right). \quad \text{M-even}$$

Phase $\theta(\omega) = \begin{cases} \pi/2 - \omega\left(\frac{M-1}{2}\right) & , \text{ if } H_r(\omega) > 0 \\ 3\pi/2 - \omega\left(\frac{M-1}{2}\right) & \text{ if } H_r(\omega) < 0. \end{cases}$

1) Determine the direct form realization of system function $H(z) = 1+2z^{-1}-3z^{-2}-4z^{-3}+5z^{-4}$.

Soln: Gn $H(z) = 1+2z^{-1}-3z^{-2}-4z^{-3}+5z^{-4}$.

$$Y(z) = X(z) + 2z^{-1}X(z) - 3z^{-2}X(z) - 4z^{-3}X(z) + 5z^{-4}X(z)$$



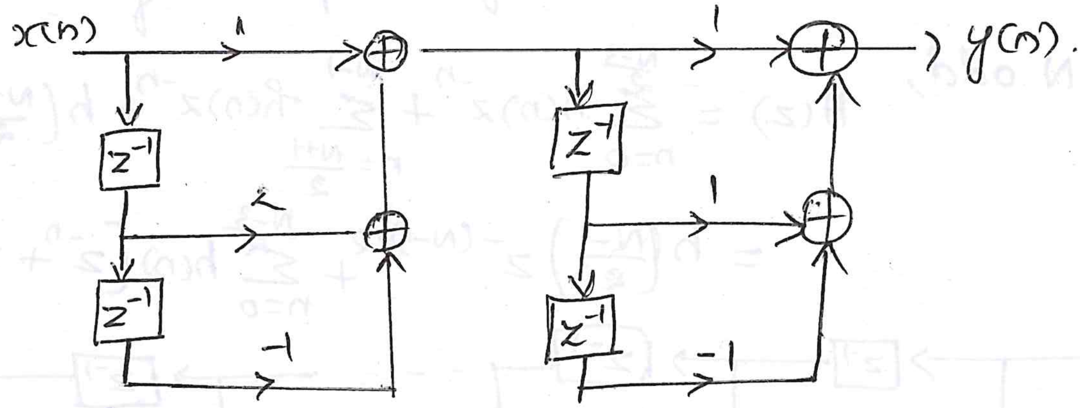
2) Obtain the cascade realization of system function $H(z) = (1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$

Soln: $H(z) = H_1(z)\,H_2(z)$

$H_1(z) = 1+2z^{-1}-z^{-2}$; $H_2(z) = 1+z^{-1}-z^{-2}$.

$H_1(z) = \dfrac{Y_1(z)}{X_1(z)}$. $Y_1(z) = X_1(z) + 2z^{-1}X_1(z) - z^{-2}X_1(z)$

$H_2(z) = \dfrac{Y_2(z)}{X_2(z)}$. $Y_2(z) = X_2(z) + z^{-1}X_2(z) - z^{-2}X_2(z)$.



3) Linear Phase Realization:

For a linear phase FIR filter

$$h(n) = h(N-1-n)$$

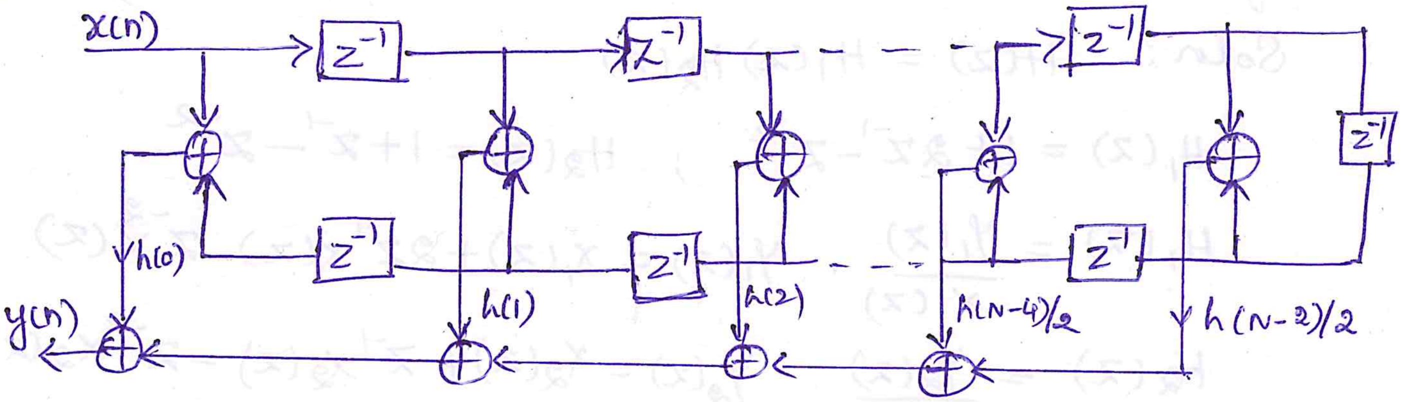
$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}.$$

For N even,
$$H(z) = \sum_{n=0}^{\frac{N-2}{2}} h(n)\, z^{-n} + \sum_{n=N/2}^{N-1} h(n)\, z^{-n}.$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)\, z^{-n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)\, z^{-(N-1-n)}.$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)\, z^{-n} + \sum_{n=0}^{\frac{N-2}{2}} h(n)\, z^{-(N-1-n)}$$

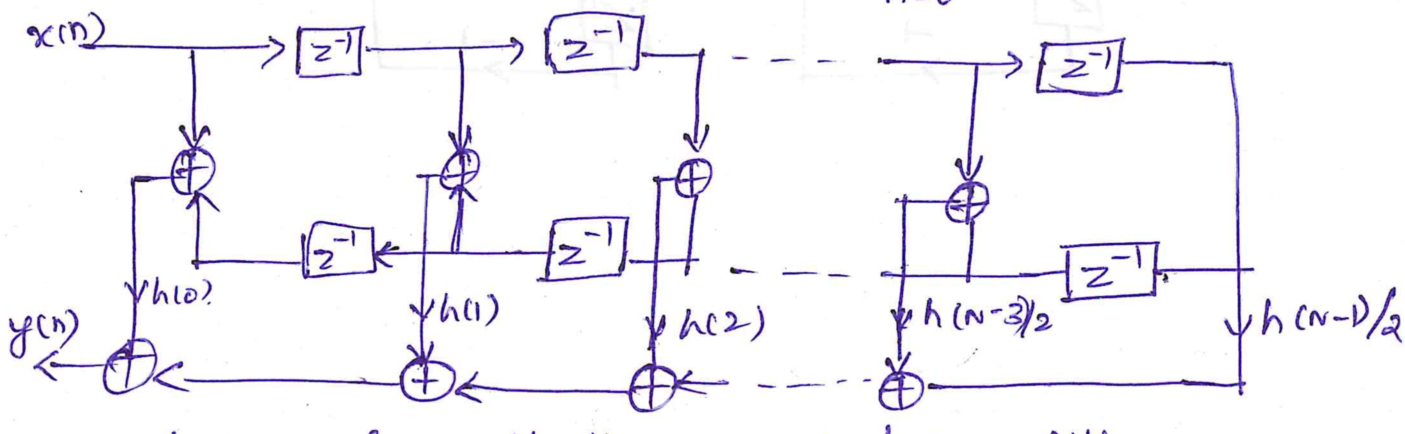
$$= \sum_{n=0}^{\frac{N-2}{2}} h(n) \left[z^{-n} + z^{-(N-1-n)}\right]$$



For N even, the no. of multipliers required is $N/2$.

For N odd,
$$H(z) = \sum_{n=0}^{\frac{N-3}{2}} h(n) z^{-n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) z^{-n} + h\left(\frac{N-1}{2}\right) z^{-(N-1)/2}$$

$$= h\left(\frac{N-1}{2}\right) z^{-(N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[z^{-n} + z^{-(N-1-n)}\right]$$



The no. of multipliers required are $\frac{N+1}{}$

4) Lattice Structure:

Consider FIR filter with system function

$$H(z) = A_m(z) = 1 + \sum_{k=1}^{m} a_m(k) z^{-k}. \quad m \geq 1$$

from this, $Y(z) = X(z)\left[1 + \sum_{k=1}^{m} a_m(k) z^{-k}\right]$

$$\therefore y(n) = x(n) + \sum_{k=1}^{m} a_m(k)\, x(n-k).$$

Interchange i/p & o/p,

$$x(n) = y(n) + \sum_{k=1}^{m} a_m(k)\, y(n-k).$$

For all-pole filter i/p $x(n) = f_N(n)$, o/p $y(n) = f_0(n)$.

For all-zero system of order $M-1$,

i/p $x(n) = f_0(n)$, o/p $y(n) = f_{m-1}(n)$.

For $m=1$, $y(n) = x(n) + a_1(1)\, x(n-1)$ —— ① from basic equ

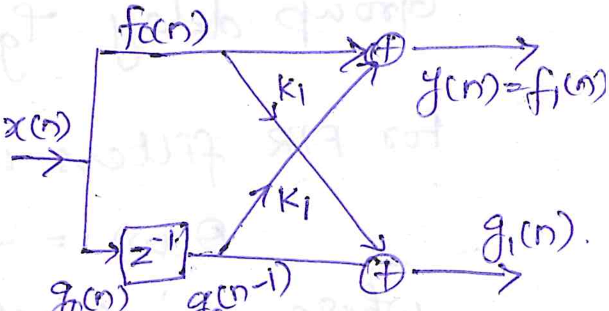
$x(n) = f_0(n) = g_0(n)$.

$y(n) = f_1(n) = f_0(n) + k_1\, g_0(n-1)$

$= x(n) + k_1\, x(n-1)$ —— ②

and $g_1(n) = k_1 f_0(n) + g_0(n-1) = k_1\, x(n) + x(n-1)$.

$a_1(0) = 1, \quad a_1(1) = k_1$



5) Polyphase realisation:

Consider FIR filter with impulse response has N coefficients.

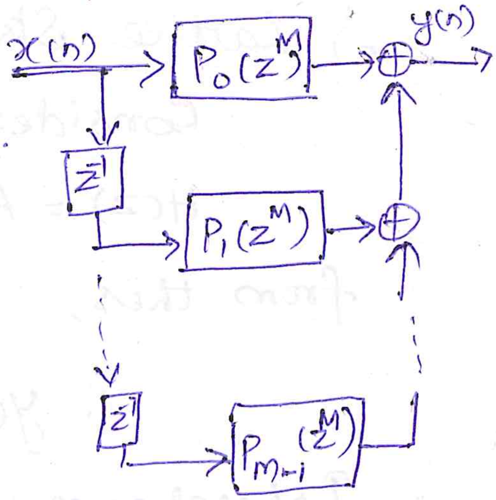
$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}.$$

H(z) can be written as $H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^M)$.

where $P_m(z^M) = \sum_{n=0}^{(N+1)/M} h(Mn+m) z^{-n} \quad 0 \leq m \leq M-1$

Replace m by $M-1-m$, then we get type 2 polyphase decomposition.

Replace m by $-m$, we obtain type 3 polyphase decomposition.



Linear Phase FIR Filter:

The transfer function of FIR causal filter is $H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$.

F.T $H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$ which is periodic in frequency with period 2π.

$$H(e^{j\omega}) = \pm |H(e^{j\omega})|\, e^{j\theta(\omega)}$$

$|H(e^{j\omega})| \rightarrow$ magnitude response

$\theta(\omega) \rightarrow$ phase response.

Phase delay $\tau_p = \frac{-\theta(\omega)}{\omega}$

Group delay $\tau_g = \frac{-d\theta(\omega)}{d\omega}$.

For FIR filters with linear phase,

$$\theta(\omega) = -\alpha\omega \qquad -\pi \le \omega \le \pi$$

where α - constant phase delay in samples.

$$\therefore \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm |H(e^{j\omega})|\, e^{j\theta(\omega)}$$

$$\sum_{n=0}^{N-1} h(n) \cos\omega n = \pm |H(e^{j\omega})| \cos\theta(\omega)$$

and $$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm |H(e^{j\omega})| \sin\theta(\omega).$$

$$\frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\sin \alpha \omega}{\cos \alpha \omega} \qquad [\because \theta(\omega) = -\alpha\omega]$$

$\sum_{n=0}^{N-1} h(n) \sin(\alpha - n)\omega = 0$, This will be zero

when $h(n) = h(N-1-n)$ & $\alpha = \frac{N-1}{2}$.

$\therefore$ FIR filters have constant phase & group delay when impulse response is symmetrical about $\alpha = \frac{N-1}{2}$

If only const. grp delay is reqd, no phase delay, we have

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j(\beta - \alpha\omega)}$$

$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| e^{j(\beta-\alpha\omega)}$ which gives

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm |H(e^{j\omega})| \cos(\beta - \alpha\omega)$$

and $-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm |H(e^{j\omega})| \sin(\beta - \alpha\omega)$.

$$\frac{-\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)}$$

From this, $\sum_{n=0}^{N-1} h(n) \sin[\beta - (\alpha - n)\omega] = 0$

If $\beta = \pi/2$, $\sum_{n=0}^{N-1} h(n) \cos(\alpha - n)\omega = 0$.

This is satisfied when $h(n) = -h(N-1-n)$ & $\alpha = \frac{N-1}{2}$

$\therefore$ FIR filters have constant group delay τ_g & not const. phase delay when the impulse response is antisymmetrical about $\alpha = \frac{N-1}{2}$.

Fourier Series Method:

The frequency response $H(e^{j\omega})$ of a system is periodic in 2π.

From fourier Series, any periodic function is represented as linear combination of complex exponentials

$\therefore$ Freq. response of FIR filter is

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}.$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n}\, d\omega.$$

Z-transform of sequence $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$.

↳ non causal filter with infinite duration.

To get FIR filter, the series can be truncated by

$$h(n) = h_d(n) \quad \text{for } |n| \le \frac{N-1}{2}$$
$$= 0 \quad \text{otherwise}$$

Then, $$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) z^{-n}.$$

$$= h\left(\frac{N-1}{2}\right) z^{-(N-1)/2} + \dots + h(1)z^{-1} + h(0) + h(-1)z + \dots + h\left[-\left(\frac{N-1}{2}\right)\right] z^{(N-1)/2}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left(h(n) z^{-n} + h(-n) z^{n}\right)$$

For a symmetrical response at $n=0$, $h(-n) = h(n)$.

$$\therefore H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{n} + z^{-n}\right]$$

For realizable filter,

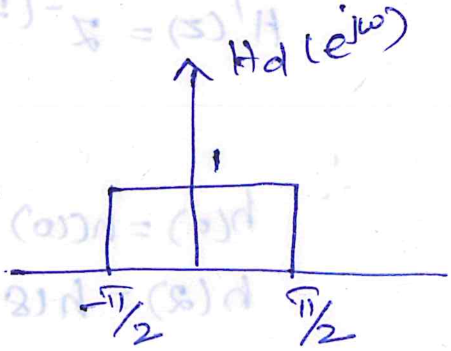
$$H'(z) = z^{-(N-1)/2} H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left(z^{n} + z^{-n}\right)\right]$$

1) Design an ideal LPF with a frequency response $H_d(e^{j\omega}) = 1$ for $-\frac{\pi}{2} \le \omega \le \frac{\pi}{2}$

$= 0$ for $\pi/2 \le |\omega| \le \pi$

Find the values of $h(n)$ for $N=11$. Plot the magnitude response.

Sol:



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi(jn)} \left| e^{j\omega n} \right|_{-\pi/2}^{\pi/2} = \frac{1}{2j\pi n}\left[e^{j\frac{\pi n}{2}} - e^{-j\frac{\pi n}{2}}\right]$$

$$= \frac{\sin\frac{\pi}{2}n}{\pi n}$$

Truncate $h_d(n)$ to 11 samples, $h(n) = \frac{\sin\frac{\pi}{2}n}{\pi n}$ for $n \le$

$= 0$ otherwise

For $n=0$, $h(n)$ is indeterminate.

So, $h(0) = \lim_{n\to 0} \frac{\sin\frac{\pi}{2}n}{\pi n} = \frac{1}{2} \lim_{n\to 0} \frac{\sin\frac{\pi}{2}n}{\frac{\pi}{2}n} = 1/2$ $\left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$

(or)

$$h(0) = h_d(0) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} d\omega$$

$$= \frac{1}{2\pi}\left.\omega\right]_{-\pi/2}^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{1}{2}.$$

For $n=1$ $h(1) = h(-1) = \frac{\sin\frac{\pi}{2}}{\pi} = \frac{1}{\pi} = 0.3183$

$h(2) = h(-2) = 0$

$h(3) = h(-3) = -0.106$

$h(4) = h(-4) = 0$

$h(5) = h(-5) = 0.06366$

The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\,(z^{n} + z^{-n})\right]$$

$$= 0.5 + 0.3183(z^{1}+z^{-1}) - 0.106(z^{3}+z^{-3}) + 0.06366(z^{5}+z^{-5})$$

Transfer function of realizable filter

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z) = z^{-5}\left[H(z)\right] = 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.106z^{-8} + 0.06$$

$\therefore$ $h(0) = h(10) = 0.06366$; $h(1) = h(9) = 0 = h(3) = h(7)$

$h(2) = h(8) = -0.106$; $h(5) = 0.5$, $h(4) = h(6) = 0.318$

Freq. response $\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n)\cos\omega n$

where $a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.5$

$a(n) = 2h\left(\frac{N-1}{2} - n\right)$

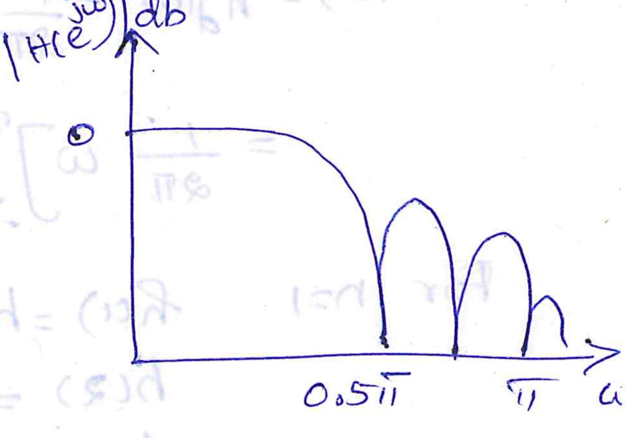
$a(1) = 2h(5-1) = 2h(4) = 0.6366$

Similarly, $a(2) = 0$, $a(3) = -0.212$, $a(4) = 0$, $a(5) = 0.127$

$$\bar{H}(e^{j\omega}) = 0.5 + 0.6366\cos\omega - 0.212\cos 3\omega + 0.127\cos 5\omega.$$

ω (in deg)	0	20	40	60	80	100
$\lvert H(e^{j\omega})\rvert_{db} = 20\log\lvert\bar{H}(e^{j\omega})\rvert$	0.4	-0.26	-0.21	0.77	-1.79	-14.56

$\omega \rightarrow$	120	140	160	180
$\lvert H(e^{j\omega})\rvert_{db}$	-20.6	-32	-30.55	-26



Filter Design Using Windows:

The frequency response $H_d(e^{j\omega})$ of a filter is periodic in frequency and canbe expanded in a fourier series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}.$$

where $h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) . e^{j\omega n} \, d\omega$ → Fourier coeff having infinite length.

One of the way of obtaining FIR filter is to truncate the infinite fourier series at $n = \pm\left(\frac{N-1}{2}\right)$ where N is the length of the desired sequence. But, abrupt truncation of Fourier series results in oscillation in the passband & stop band. These oscillations due to slow convergence of Fourier series & this effect is Gibbs phenomenon.

To reduce these oscillations, Fourier coeff. of the filter are modified by multiplying infinite impulse response with a weighing sequence $w(n)$ is called window.

$$w(n) = w(-n) \neq 0 \quad \text{for } |n| \leq \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{for } |n| > \left(\frac{N-1}{2}\right)$$

After multiplying $w(n)$ with $h_d(n)$, we get a finite duration sequence $h(n)$ that satifies magnitude response

$$h(n) = h_d(n)\, w(n) \quad \text{for } |n| \leq \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{for } |n| > \left(\frac{N-1}{2}\right)$$

Frequency response $H(e^{j\omega})$ is obtained by convolution of $H_d(e^{j\omega})$ & $W(e^{j\omega})$ given by

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$

$$= H_d(e^{j\omega}) * W(e^{j\omega}) \rightarrow \text{Periodic convolution}$$

characteristics of window:

1) The central lobe of frequency response contains most of the energy & should be narrow.
2) The highest side lobe level of frequency response is small.
3) The side lobes of frequency response should decrease in energy as $\omega \rightarrow \pi$.

Rectangular Window:

Rectangular window sequence is given by,

$$\omega_R(n) = 1 \quad \text{for } -(N-1)/2 \leq n \leq (N-1)/2$$
$$= 0 \quad \text{otherwise}.$$

Spectrum of rectangular window is

$$W_R(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\omega n}$$

$$= e^{j\omega(N-1)/2} + \cdots + e^{j\omega} + 1 + e^{-j\omega} + \cdots + e^{-j\omega(N-1)/2}$$

$$= e^{j\omega(N-1)/2}\left[1 + e^{-j\omega} + \cdots + e^{-j\omega(N-1)}\right]$$

$$= e^{j\omega(N-1)/2}\left[\frac{1-e^{j\omega N}}{1-e^{-j\omega}}\right] \qquad 1+a+a^2+\cdots a^{N-1} = \frac{1-a^N}{1-a}$$

$$= \frac{e^{j\omega N/2}\,(1-e^{-j\omega N})}{e^{j\omega/2}\,(1-e^{-j\omega})} = \frac{\sin \omega N/2}{\sin \omega/2}.$$

The frequency response for ω between $\frac{2\pi}{N}$ & $-\frac{2\pi}{N}$ is called main lobe & other lobes are side lobes. (ie) $\omega < -\frac{2\pi}{N}$ or $\omega > \frac{2\pi}{N}$.

Main lobe width $= \frac{4\pi}{N}$.

Higher side lobe = 22% of main lobe (or) -13db relative to max. value at $\omega=0$

Finite impulse response $h(n) = h_d(n) \omega_R(n)$.

Frequency response of truncated filter is

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) W_R(e^{j(\omega-\theta)}) d\theta.$$

Hanning Window :

The Raised cosine window is of the form

$$\omega_\alpha(n) = \alpha + (1-\alpha)\cos\frac{2\pi n}{N-1} \quad \text{for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$
$$= 0 \quad \text{otherwise}.$$

Hanning window sequence can be obtained by substituting $\alpha = 0.5$

$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$
$$= 0 \quad \text{otherwise}$$

The frequency response of Hanning window is

$$W_{Hn}(e^{j\omega}) = 0.5\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.25\frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))}$$
$$+ 0.25\frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}$$

Main lobe width $\rightarrow$ Twice of rectangular window.

Magnitude of sidelobe = -31 db (ie) 18 db less than rectangular window

Min. stop band attenuation = 44 db (ie) 23 db less than rectangular window

Hamming Window:

This window sequence is obtained by, $\alpha = 0.54$

$$w_H(n) = 0.54 + 0.46 \cos(2\pi n/N-1) \quad \text{for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$
$$= 0 \quad \text{otherwise}$$

Frequency response of Hamming window is

$$W_H(e^{j\omega}) = 0.54 \frac{\sin \omega N/2}{\sin \omega/2} + 0.23 \frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))}$$
$$+ 0.23 \frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}$$

Peak sidelobe level = 41 db from main lobe peak.

Sidelobe peak (first) = -53 db.

Hamming window generates less oscillation in the sidelobes than Hanning window.

Ex: Design an ideal HPF with a frequency response $H(e^{j\omega}) = 1$ for $\frac{\pi}{4} \le |\omega| \le \pi$

$= 0$ for $|\omega| \le \frac{\pi}{4}$.

Find the values of $h(n)$ for $N=11$. Find $H(z)$. Plot magnitude response.

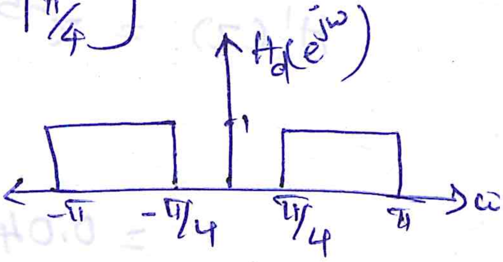
Repeat the same using a) Hanning window b) Hamming window also.

Soln:

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$
$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega \right]$$

$$h_d(n) = \frac{1}{2\pi j n}\left[e^{j\omega n}\Big|_{-\pi}^{-\pi/4} + e^{j\omega n}\Big|_{\pi/4}^{\pi}\right]$$

$$= \frac{1}{\pi n(2j)}\left[e^{-j\pi n/4} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi n/4}\right]$$



$$= \frac{1}{\pi n}\left[\sin \pi n - \sin\frac{\pi}{4}n\right] \quad -\infty \le n \le \infty$$

Truncating $h_d(n)$ to 11 samples, we have

$$h(n) = h_d(n) \text{ for } |n| \le 5$$
$$= 0 \quad \text{otherwise}$$

For $n=0$, $h(0) = \lim_{n\to 0} \frac{\sin \pi n}{\pi n} - \lim_{n \to 0} \frac{\sin \frac{\pi}{4} n}{\pi n}$

$$= 1 - \frac{1}{4} = 0.75 \quad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1,\ \lim_{\theta\to 0}\frac{\sin n\theta}{\theta} = n\right]$$

$$h(1) = h(-1) = \frac{\sin \pi - \sin \pi/4}{\pi} = -0.225$$

$$h(2) = h(-2) = \frac{\sin 2\pi - \sin \pi/2}{2\pi} = -0.159$$

$$h(3) = h(-3) = \frac{\sin 3\pi - \sin 3\pi/4}{3\pi} = -0.075$$

$$h(4) = h(-4) = \frac{\sin 4\pi - \sin\pi}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} = 0.045$$

The transfer function of filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}}\left[h(n)(z^n + z^{-n})\right]$$

$$= 0.75 + \sum_{n=1}^{5}\left[h(n)(z^n+z^{-n})\right]$$

$$= 0.75 - 0.225(z+z^{-1}) - 0.159(z^2+z^{-2}) - 0.075(z^3+z^{-3}) + 0.045(z^5+z^{-5}).$$

The transfer function of realizable filter is

$$H'(z) = z^{-5}H(z) = z^{-5}\left[0.75 - 0.225(z+z^{-1}) - 0.159(z^2+z^{-2}) - 0.075(z^3+z^{-3}) + 0.045(z^5+z^{-5})\right]$$

$$= 0.045 - 0.075z^{-2} - 0.159z^{-3} - 0.225z^{-4} + 0.75z^{-5} - 0.225z^{-6} - 0.159z^{-7} - 0.075z^{-8} + 0.045z^{-10}.$$

Filter coefficients of causal filter are

$h(0) = h(10) = 0.045$, $h(1) = h(9) = 0$; $h(2) = h(8) = -0.075$

$h(3) = h(7) = -0.159$, $h(4) = h(6) = -0.225$; $h(5) = 0.75$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n \quad \text{where}$$

$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$; $\quad a(n) = 2h\left(\frac{N-1}{2} - n\right)$

$a(1) = 2h(5-1) = 2h(4) = -0.45.$

$a(2) = 2h(5-2) = 2h(3) = -0.318$

$a(3) = 2h(5-3) = 2h(2) = -0.15$

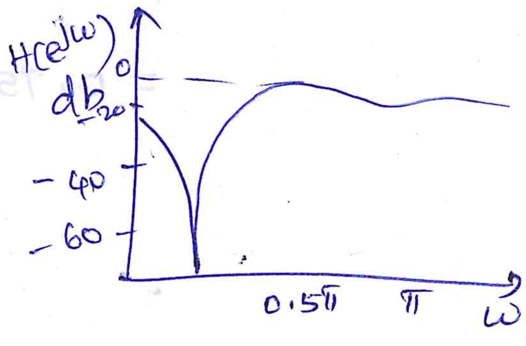
$a(4) = 2h(5-4) = 2h(1) = 0$

$a(5) = 2h(5-5) = 2h(0) = 0.09$

$$\bar{H}(e^{j\omega}) = a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega.$$

$$= 0.75 - 0.45\cos\omega - 0.318\cos 2\omega - 0.15\cos 3\omega + 0.09\cos 5\omega$$

ω	0	20	40	60	80	100
$H(e^{j\omega})$	-0.08	-0.0086	0.34	0.88	1.11	0.98
$H(e^{j\omega})_{db}$	-22	-41.3	-9.36	-1.1	0.95	-0.132



a) Hanning Window:

$w_{Hn}(n) = 0.5 + 0.5 \cos \frac{2\pi n}{N-1}$ for $\frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2}$

$= 0$ otherwise.

For N=11, $w_{Hn}(n) = 0.5 + 0.5 \cos \frac{\pi n}{5}$ $\quad -5 \le n \le 5$

$= 0$ otherwise.

$w_{Hn}(0) = 0.5 + 0.5 = 1$

$w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5 \cos \frac{\pi}{5} = 0.9045$

$w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5 \cos \frac{2\pi}{5} = 0.655$

$w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5 \cos \frac{3\pi}{5} = 0.345$

$w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5 \cos \frac{4\pi}{5} = 0.0945$

$w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5 \cos \pi = 0$

Filter coefficients using Hanning Window are

$h(n) = h_d(n)\, w_{Hn}(n)$ for $-5 \le n \le 5$

$= 0$ otherwise

$h(0) = h_d(0)\, w_{Hn}(0) = 0.75\,(1) = 0.75$

$h(1) = h(-1) = h_d(1)\, w_{Hn}(1) = (-0.225)(0.905) = -0.204$

$h(2) = h(-2) = h_d(2)\, w_{Hn}(2) = (-0.159)(0.655) = -0.104$

$h(3) = h(-3) = h_d(3)\, w_{Hn}(3) = (-0.075)(0.345) = -0.026$

$h(4) = h(-4) = h_d(4)\, w_{Hn}(4) = (0)(0.0945) = 0$

$h(5) = h(-5) = h_d(5)\, w_{Hn}(5) = (0.045)(0) = 0.$

The transfer function of the filter is

$H(z) = h(0) + \sum_{n=1}^{5} h(n)\,(z^{-n} + z^{n})$

$= 0.75 - 0.204\,(z + z^{-1}) - 0.104\,(z^{2} + z^{-2}) - 0.026\,(z^{3} + z^{-3})$

The transfer function of realizable filter

$$H'(z) = z^{-5}H(z) = -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6} - 0.104z^{-7} - 0.026z^{-8}.$$

The causal filter coefficients are

$h(0) = h(1) = h(9) = h(10) = 0.$

$h(2) = h(8) = -0.026$; $h(3) = h(7) = -0.104.$

$h(4) = h(6) = -0.204$; $h(5) = 0.75.$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

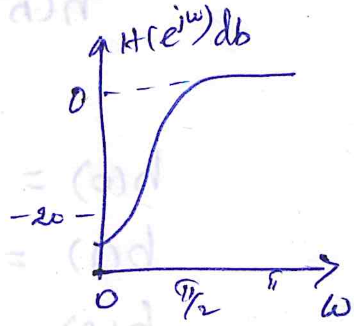
$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$; $a(n) = 2h\left(\frac{N-1}{2} - n\right)$

$a(1) = 2h(5-1) = 2h(4) = -0.408$

Similarly, $a(2) = -0.208$, $a(3) = -0.052$, $a(4) = 0$
$a(5) = 0.$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052\cos 3\omega$$

ω	0	30	60	90	120	150
$\bar{H}(e^{j\omega})$	0.082	0.292	0.702	0.96	1.006	1
$\bar{H}(e^{j\omega})_{db}$	-21.72	-10.67	-3.07	-0.3726	0.052	0



b) <u>Hamming Window;</u>

$$\omega_H(n) = 0.54 + 0.46\cos\frac{2\pi n}{N-1} \quad \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{otherwise.}$$

$\therefore \omega_H(0) = 0.54 + 0.46 = 1.$; $\omega_H(3) = \omega_H(-3) = 0.398$

$\omega_H(1) = \omega_H(-1) = 0.912$; $\omega_H(4) = \omega_H(-4) = 0.1678$

$\omega_H(2) = \omega_H(-2) = 0.682$; $\omega_H(5) = \omega_H(-5) = 0.08.$

The filter coefficients $h(n) = h_d(n)\,\omega_H(n).$

$h(0) = 0.75$; $h(1) = h(-1) = -0.2052$, $h(2) = h(-2) = -0.108$

$h(3) = h(-3) = -0.03$; $h(4) = h(-4) = 0$; $h(5) = h(-5) = 0.0036.$

The transfer function of filter is

$$H(z) = h(0) + \sum_{n=1}^{5}\left[h(n)(z^{-n}+z^{n})\right]$$

$$= 0.75 - 0.2052(z^{-1}+z) - 0.1084(z^{-2}+z^{2}) - 0.03(z^{-3}+z^{3}) + 0.0036(z^{-5}+z^{5})$$

The transfer function of realizable filter is

$$H'(z) = z^{-5}H(z) = 0.0036 - 0.03z^{-2} - 0.1084z^{-3} - 0.2052z^{-4} + 0.75z^{-5} - 0.2052z^{-6} - 0.1084z^{-7} - 0.03z^{-8} + 0.0036z^{-10}$$

Filter coefficients of causal filter are

$h(0) = h(10) = 0.0036$; $h(1) = h(9) = 0$; $h(2) = h(8) = -0.03$

$h(3) = h(7) = -0.1084$; $h(4) = h(6) = -0.2052$; $h(5) = 0.75$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

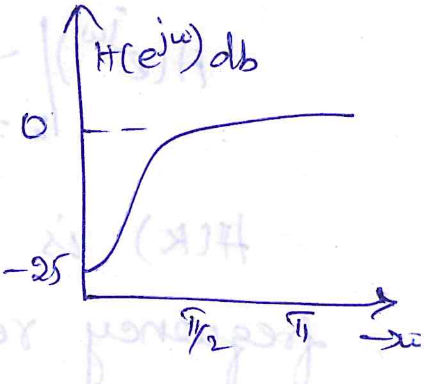
$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$; $a(n) = 2h\left(\frac{N-1}{2} - n\right)$

$\therefore$ $a(1) = -0.4104$, $a(2) = -0.2168$; $a(3) = -0.06$

$a(4) = 0$, $a(5) = 0.0072$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.4104\cos\omega - 0.2168\cos2\omega - 0.06\cos3\omega + 0.0072\cos5\omega$$

ω	0	30	60	90
$\bar{H}(e^{j\omega})$	0.07	0.28	0.7168	0.9668
$\bar{H}(e^{j\omega})_{db}$	-23.1	-11	-2.87	-0.29



ω	120	150	180
$H(e^{j\omega})$	1	1.003	1.0108
$H(e^{j\omega})_{db}$	0	0.028	0.093.

Frequency Sampling Method:

Let $h(n)$ - filter coefficients

$H(k)$ - DFT of $h(n)$, then

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j2\pi kn/N}, \quad n = 0, 1 \cdots N-1$$

$$H(k) = \sum_{k=0}^{N-1} h(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1 \cdots N-1.$$

$$H(k) = H(z)\Big|_{z = e^{j2\pi k/N}}.$$

The transfer function $H(z)$ of an FIR filter with impulse response $H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$.

$$H(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j2\pi kn/N}\right] z^{-n}$$

$$= \sum_{k=0}^{N-1}\frac{H(k)}{N}\sum_{n=0}^{N-1} H(k)\left(e^{j2\pi k/N} z^{-1}\right)^{n}.$$

$$= \sum_{k=0}^{N-1}\frac{H(k)}{N}\,\frac{1-\left(e^{j2\pi k/N} z^{-1}\right)^{N}}{1-e^{j2\pi k/N} z^{-1}}.$$

$$= \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-e^{j2\pi k/N} z^{-1}}.$$

$$H(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} = H\left(e^{j2\pi kn)N}\right) = H(k).$$

$H(k)$ is k^{th} DFT component obtained by sampling frequency response $H(e^{j\omega})$. This approach for designing FIR filter is called frequency sampling method.

i) Frequency Sampling realization:

$$H(z) = \frac{1-z^{-N}}{N} \sum_{K=0}^{N-1} G_K(z).$$

where $G_K(z) = \dfrac{H(K)}{1-e^{j2\pi K/N} z^{-1}}$ $\quad 0 \le K \le N-1.$

is the transfer function of first order FIR filter where poles lie on the unit circle at equidistant points.

ii) Frequency response:

It is obtained by setting $z = e^{j\omega}$

$$H(e^{j\omega}) = \frac{1-e^{-j\omega N}}{N} \sum_{K=0}^{N-1} \frac{H(K)}{1-e^{j2\pi K/N} e^{-j\omega}}$$

$$= \frac{e^{-j\omega N/2}\left(e^{j\omega N/2} - e^{-j\omega N/2}\right)}{N} \sum_{K=0}^{N-1} \frac{H(K)}{1-e^{-j(\omega - 2\pi K/N)}}$$

$$= \frac{e^{-j\omega N/2}}{N} \sum_{K=0}^{N-1} \frac{H(K)\left(e^{j\omega N/2} - e^{-j\omega N/2}\right)}{e^{-j(\omega/2 - \pi K/N)}\left[e^{j(\omega/2 - \frac{\pi K}{N})} - e^{-j(\omega/2 - \frac{\pi K}{N})}\right]}$$

$$= \frac{e^{-j\omega(N-1)/2}}{N} \sum_{K=0}^{N-1} \frac{H(K)\, e^{-j\pi K/N} \sin \omega N/2}{\sin(\omega/2 - \pi K/N)}.$$

$$= \frac{e^{-j\omega(N-1)/2}}{N} \sum_{K=0}^{N-1} \frac{H(K)(-1)^K e^{-j\pi K/N} \sin N(\omega/2 - \frac{K\pi}{N})}{\sin(\omega/2 - \pi K/N)}$$

Type I design: Initial pt of freq. Sample is at $\boxed{\omega = 0}$, spacing $\frac{2\pi}{N}$ bet. 2 pts

Freq. Samples $H(K) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}K}$. $\quad K = 0, 1 \ldots N-1$

$$H(K) = |H(K)|\, e^{j\theta(K)}.$$

For linear phase $\theta(K) = -\alpha\omega\Big|_{\omega = \frac{2\pi}{N}K}$. $\quad K = 0, 1, \ldots N-1$

$$= -\left(\frac{N-1}{2}\right)\frac{2\pi}{N}K = -\left(\frac{N-1}{N}\right)\pi K$$

Filter coefficients $h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{j2\pi nk/N}$ $n=0,1\cdots N-1$

For N odd or even $H(N-k) = H^*(k)$. $k=0,1\cdots N-1$

for N even $H\left(\frac{N}{2}\right) = 0$

$|H(k)| = |H(N-k)|$; $\theta(k) = -\theta(N-k)$.

$\theta(N-k) = -\left(\frac{N-1}{N}\right)\pi(N-k) = -(N-1)\pi + \left(\frac{N-1}{N}\right)\pi k$.

For N odd, $\theta(k) = -\left(\frac{N-1}{N}\right)\pi k$, $k=0,1,\cdots \frac{N-1}{2}$

$= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k$, $k=\frac{N+1}{2},\cdots N-1$.

For N even, $\theta(k) = -\left(\frac{N-1}{N}\right)\pi k$, $k=0,1\cdots \frac{N}{2}-1$

$= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k$, $k=\frac{N}{2}+1,\cdots N-1$

$= 0$ $k=N/2$.

For N odd, $H(k) = |H(k)|\, \underbrace{e^{-j(N-1)\pi k/N}}_{e^{j\theta(k)}}$, $k=0,1\cdots \frac{N-1}{2}$

$= |H(k)|\, e^{j(N-1)\pi - (N-1)\pi k/N}$, $k=\frac{N+1}{2}\cdots N-1$

For N even, $H(k) = |H(k)|\, e^{-j(N-1)\pi k/N}$, $k=0,1\cdots \frac{N}{2}-1$

$= |H(k)|\, e^{j[(N-1)\pi - (N-1)\pi k/N]}$, $k=\frac{N}{2}+1,\cdots N-1$

$= 0$, $k=N/2$.

If the filter is to be linear phase,

$h(n) = h(N-1-n)$.

$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{(N-1)/2} Re\left\{\left[H(k)e^{j2\pi kn/N}\right]\right\}\right\}$ N-odd

and

$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} Re\left[H(k)\, e^{j2\pi kn/N}\right]\right\}$ N-even

Type 2 design:

Frequency samples $H(K) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}(K+\frac{1}{2})}$.

$H(K) = H_d\left(e^{j\pi(2k+1)/N}\right)$, $k = 0, 1 - - - N-1$

The initial point of freq. samples is located at $\boxed{\omega = \frac{\pi}{N}}$,

spacing between two points is $\frac{2\pi}{N}$.

Filter coefficients $h(n) = \sum_{k=0}^{N-1} H(k) e^{j2\pi kn/N}$, $k = 0, 1 --- N-1$

The condition that $h(n)$ be real is,

for N odd, $H(N-k-1) = H^*(k)$, $k = 0, 1 \cdots \frac{N-1}{2} - 1$

$H\left(\frac{N-1}{2}\right) = 0$

for N even, $H(N-k-1) = H^*(k)$ $\quad k = 0, 1, \cdots \frac{N}{2} - 1$

When these conditions are satisfied, filter coefficients

for N odd, $h(n) = \frac{2}{N} \sum_{k=0}^{\frac{N-3}{2}} Re\left[H(k) e^{j\pi n(2k+1)/N}\right]$

for N even, $h(n) = \frac{2}{N} \sum_{k=0}^{\frac{N}{2}-1} Re\left[H(k) e^{jn\pi(2k+1)/N}\right]$

Ex.1: Determine the filter coefficients $h(n)$ obtained by sampling $\theta(k)$. Apr. 2019] (Apr. 2018)

$|H(k)|$

$H_d(e^{j\omega}) = 1 \cdot e^{-j(N-1)\omega/2}$ $\quad 0 \le |\omega| \le \pi/2$

$= 0$ $\quad \pi/2 \le |\omega| \le \pi$ for N=

Soln: Given $N = 7$,

$H(K) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$ substitute k. $K = 0, 1, 2 --- 6$.

find $\omega \le \pi/2$, put $|H(k)| = 1$

$|H(k)| = 1$ for $k = 0, 1, 6$ $\quad$ $k=0$ $\omega = 0$

0 for $K = 2, 3, 4, 5$ $\quad$ $k=1$ $\omega = 51°$

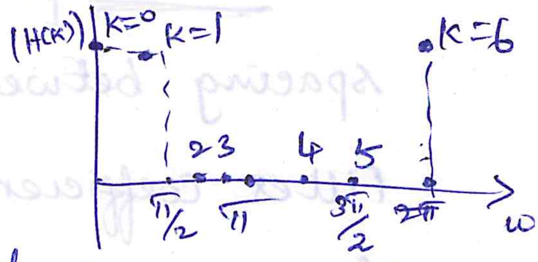
$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k = -\frac{6}{7}\pi k \quad \text{for } k = 0,1,2,3$$

$$= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k = 6\pi - \frac{6\pi k}{7} = \frac{6\pi}{7}(7-k) \quad \text{for } k = 4,5,6$$

$$H(k) = e^{-j6\pi k/7} \qquad k = 0,1$$

$$= 0 \qquad k = 2,3,4,5$$

$$= e^{-j6\pi(k-7)/7} \quad \text{for } k = 6.$$



The filter coefficients for N odd are

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k)e^{j2\pi kn/7}\right]\right\} \quad n = 0,1,\dots N-1$$

$$= \frac{1}{7}\left\{1 + 2\,\mathrm{Re}\left(e^{-j6\pi/7}e^{j2\pi n/7}\right)\right\}$$

$$= \frac{1}{7}\left\{1 + 2\,\mathrm{Re}\left(e^{j2\pi(n-3)/7}\right)\right\}$$

$$= \frac{1}{7}\left\{1 + 2\cos\frac{2\pi}{7}(n-3)\right\}$$

$h(n) = h(N-1-n)$

$$h(0) = h(6) = \frac{1}{7}\left(1 + 2\cos\frac{6\pi}{7}\right) = -0.11456$$

$$h(1) = h(5) = \frac{1}{7}\left(1 + 2\cos\frac{4\pi}{7}\right) = 0.07928$$

$$h(2) = h(4) = \frac{1}{7}\left(1 + 2\cos\frac{2\pi}{7}\right) = 0.321$$

$$h(3) = \frac{1}{7}(1+2) = 0.42857$$

Ex.2: Find the coefficients of a linear phase FIR filter of length $M=15$ has a symmetric unit sample response and a frequency response that satisfies the condition. (Apr. 2017)

$$H\left(\frac{2\pi k}{15}\right) = 1 \qquad k = 0,1,2,3$$

$$= 0 \qquad k = 4,5,6,7$$

$H(k) = H(N-k)$ $\quad H(1) = H(14)$

$H(0) = H(15-0)$

Solu: $|H(k)| = 1$ for $0 \le k \le 3$ & $12 \le k \le 14$

$= 0$ for $4 \le k \le 11$.

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k = \frac{-14}{15}\pi k \qquad 0 \le k \le 7$$

$$= 14\pi - \frac{14\pi k}{15} \quad \text{for } 8 \le k \le 14.$$

$$H(k) = e^{-j14\pi k/15} \quad \text{for } k = 0, 1, 2, 3$$

$$= 0 \quad \text{for } 4 \le k \le 11$$

$$= e^{-j14\pi(k-15)/15} \quad \text{for } 12 \le k \le 14.$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left(H(k)\, e^{j2\pi nk/15}\right)\right]$$

$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{7} Re\left(e^{-j14\pi k/15}\, e^{j2\pi nk/15}\right)\right]$$

$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} \cos\frac{2\pi k(7-n)}{15}\right]$$

$$= \frac{1}{15}\left[1 + 2\cos\frac{2\pi(7-n)}{15} + 2\cos\frac{4\pi(7-n)}{15} + 2\cos\frac{6\pi(7-n)}{15}\right]$$

$h(0) = h(14) = -0.05$, $\quad h(1) = h(13) = 0.041$

$h(4) = h(10) = -0.1078$, $\quad h(2) = h(12) = 0.066$;

$h(3) = h(11) = -0.0365$, $\quad h(5) = h(9) = 0.034$.

$h(6) = h(8) = 0.3188$, $\quad h(7) = 0.466$

Finite word length effects in digital filters:

[Apr, 2019]

DSP algorithms are realized with digital hardware. The numbers and coefficients are stored in finite word-length registers. So, coefficients are quantized by truncation or round off when they are stored.

Errors:

1) Input quantization error:

The conversion of a continuous time input signal into digital value produces an error. (ie) i/p error. This arises due to representation of i/p signal by a fixed no. of digits in ADC process.

2) Product quantisation error:

It arises at o/p of a multiplier. Multiplication of 'b' bit with 'b' bit coefficient results in 2b bits. Since, 'b' bit register is used, o/p is rounded to b bits which produce error.

3) Coefficient quantisation error:

Filter coefficients are computed to infinite precision in theory. If they are quantized, frequency response differs from desired response that leads to instability.

UNIT IV. FINITE WORDLENGTH EFFECTS

Introduction:

A number 'N' is represented by a finite series $N = \sum_{i=n_1}^{m_2} c_i r^i$ where r is called as radix.

If $r = 10$ indicates decimal

$r = 2$ indicates binary etc.

$$30.285 = \sum_{i=-3}^{1} c_i 10^i$$
$$= 3\times10^1 + 0\times10^0 + 2\times10^{-1} + 8\times10^{-2} + 5\times10^{-3}$$

Fixed point representation:

In this, position of binary point is fixed.

Bit to right → fractional part of number

left → integer part.

Negative numbers are represented in different forms:

1) Sign-magnitude form
2) One's complement form
3) Two's Complement form.

Sign-magnitude form:

→ MSB is set to 1 for negative sign.

ex: $-1.75 \rightarrow 11.110000$

$1.75 \rightarrow 01.110000$

One's complement form:

→ positive no. is in sign-magnitude notation

→ Negative no. obtained by complementing all the bits of positive number.

ex: $(0.875)_{10} = (0.111000)_2$

$(-0.875) = (1.000111)_2$

Two's complement form:

→ positive numbers are represented in sign-magnitude and one's complement.

→ Negative numbers are by complementing all the bits of positive no. and adding one to LSB.

ex: $(0.875)_{10} = (0.111000)_2$

$$\begin{array}{rl} & \downarrow\ \downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \\ & 1.000111 \quad \text{(Complement each bit)} \\ + & 0.000001 \quad \text{(Add 1)} \\ \hline (-0.875)_{10} = & 1.001000 \end{array}$$

Floating point Representation:

→ Positive no. is represented as $F = 2^c . M$

M - Mantissa, is a fraction that $\frac{1}{2} \leq M \leq 1$

c - Exponent, either positive or negative.

ex: $4.5 \rightarrow 2^3 \times 0.5625 = 2^{011} \times 0.1001$

→ Negative no. is considered by representing mantissa as a fixed point number.

$F_1 = 2^{c_1} \times M_1$, $F_2 = 2^{c_2} \times M_2$

$\therefore$ Product $F_3 = (M_1 \times M_2)\, 2^{c_1 + c_2}$.

Fixed Point	Floating point
Fast Operation	Slow operation.
Relatively economical	More expensive
Small dynamic range	Increased dynamic range.
Overflow occurs.	Overflow doesn't arise
Used in small computers	Used in large computers.

ADC & Quantization:

For most applications, i/p signal is continuous. This signal is converted into digital by ADC.

$$\xrightarrow{x(t)} \boxed{\text{Sampler}} \xrightarrow{x(n)} \boxed{\text{Quantizer}} \longrightarrow x_q(n)$$

Block diagram of ADC.

In fig, $x(t)$ is sampled at regular intervals $t = nT$ where $n = 0, 1, 2 \ldots$ to create sequence $x(n)$. This is done by sampler. $x(n)$ is expressed by finite no. of bits giving sequence $x_q(n)$.

Difference signal $e(n) = x_q(n) - x(n)$ is quantization noise (or) A/D conversion noise.

Consider sine signal varying between +1 & −1 having dynamic range 2. If ADC is used to convert sine signal, it employs $(b+1)$ bits including sign bit, No. of levels for quantizing $x(n)$ is 2^{b+1}.

Interval between successive levels $q = \frac{2}{2^{b+1}} = 2^{-b}$.

q - quantisation step size.

Methods of quantization:

1) Truncation
2) Rounding.

Truncation:

→ Process of discarding all bits less significant than LSB that is retained.

ex: 0.00110011 to 0.0011
(8 bits) (4 bits)

Rounding:

→ Rounding of a number 'b' bits is done by choosing rounded result as 'b' bit no. closest to original no. unrounded.

ex: 0.11010 rounded to 3 bits as 0.110 or 0.111

Truncation and Rounding Errors:

In truncation, no. is approximated by the nearest level that doesn't exceed it.

In this, error $x_T - x$ is negative or zero where x_T is truncation value of x, it is assumed $|x| \leq 0$.

Error made by truncating a no. to 'b' bits following binary points satisfy inequality,

$$0 \geq x_T - x > -2^{-b}.$$

ex: Decimal no. $0.12890625 \rightarrow (0.00100001)_2$

Now, truncate binary number to 4 bits $\rightarrow x_T = (0.0010)$
$= (0.125)_{10}$

$\therefore$ error $(x_T - x) = -0.00390625$.

(ie) greater than $-2^{-b} = -2^{-4} = -0.0625$

Consider <u>two's complement</u> representation, magnitude of negative no. is $x = 1 - \sum_{i=1}^{b} c_i 2^{-i}$.

Truncate it to N bits then $x_T = 1 - \sum_{i=1}^{N} c_i 2^{-i}$.

Change in magnitude $x_T - x = \sum_{i=1}^{b} c_i 2^{-i} - \sum_{i=1}^{N} c_i 2^{-i}$.

$$= \sum_{i=N}^{b} c_i 2^{-i} \geq 0$$

Due to truncation, change in magnitude is +ve, then the error is -ve & satisfy inequality

$$0 \geq x_T - x > -2^{-b}.$$

For <u>one's complement</u> representation, magnitude of negative no. with 'b' bits is

$$x = 1 - \sum_{i=1}^{b} c_i 2^{-i} - 2^{-b}$$

When no. is truncated to N bits, then

$$x_T = 1 - \sum_{i=1}^{N} c_i 2^{-i} - 2^{-N}.$$

Change in magnitude due to truncation is

$$x_T - x = \sum_{i=N}^{b} c_i 2^{-i} - (2^{-N} - 2^{-b}) < 0$$

$\therefore$ Magnitude decreases with truncation which implies that error is positive and satisfy inequality $0 \leq x_T - x < 2^{-b}$.

In <u>floating point</u> systems, effect of truncation is visible only in mantissa.

Let mantissa is truncated to N bits

If $x = 2^c \cdot M$ then $x_T = 2^c \cdot M_T$

Error $e = x_T - x = 2^c (M_T - M)$

Two's complement representation of mantissa,

$$0 \geq M_T - M > -2^{-b}$$

$$0 \geq e > -2^{-b} 2^c .$$

Relative error $\epsilon = \dfrac{x_T - x}{x} = \dfrac{e}{x}$.

$$\therefore\ 0 \geq \epsilon x > -2^{-b} . 2^c \quad \text{(or)} \quad 0 \geq \epsilon 2^c M > -2^{-b} 2^c .$$

(or) $\quad 0 \geq \epsilon M > -2^{-b}$.

If $M = 1/2$, the relative error is maximum.

$$\therefore\ 0 \geq \epsilon > -2 . 2^{-b} .$$

If $M = -1/2$, relative error range is

$$0 \leq \epsilon < 2 . 2^{-b}$$

In One's complement representation, error for truncation of positive values of mantissa is

$$0 \geq M_T - M > -2^{-b} \quad \text{(or)} \quad 0 \geq e > -2^{-b} . 2^c$$

with $e = \epsilon x = \epsilon 2^c . M$ and $M = 1/2$.

Max. range of relative error for positive M is

$$0 \geq \epsilon > -2 . 2^{-b}$$

For negative mantissa values, the error is

$$0 \leq M_T - M < 2^{-b} \quad \text{(or)} \quad 0 \leq e < 2^c . 2^{b} \quad \text{with } M = -1/2$$

Max. range of relative error for negative M is

$0 \geq \epsilon > -2 . 2^{-b}$ which is same as positive M.

b) Rounding error:

In fixed point arithmetic, error due to rounding a number to b bits produces an error $e = x_T - x$ which satisfy inequality

$$\frac{-2^{-b}}{2} \le x_T - x \le \frac{2^{-b}}{2}.$$

With rounding, In floating point arithmetic, only mantissa is affected by quantisation.

If $x = M.2^c$, $x_T = M_T\, 2^c$ then

$$e = x_T - x = (M_T - M)2^c.$$

For rounding, $\frac{-2^{-b}}{2} \le M_T - M \le \frac{2^{-b}}{2}$.

$$+2^c \frac{-2^{-b}}{2} \le x_T - x \le 2^c \frac{2^{-b}}{2} \quad \text{(or)}$$

$$+2^c \frac{-2^{-b}}{2} \le \epsilon x \le 2^c \frac{2^{-b}}{2}$$

But $x = 2^c.M$ then $+2^c \frac{-2^{-b}}{2} \le \epsilon\, 2^c.M \le 2^c \frac{2^{-b}}{2}$

which gives $-\frac{2^{-b}}{2} \le \epsilon.M \le \frac{2^{-b}}{2}$

The mantissa satisfies $\frac{1}{2} \le M < 1$.

If $M = \frac{1}{2}$, max. range of relative error $-2^{-b} \le \epsilon < 2^{-b}$

Quantization error ranges

Types of quantisation	Type of arithmetic	Fixed point no. range	Floating point no. relative error range
Rounding	Sign-magnitude, One's complement, Two's "	$\frac{-2^{-b}}{2} \le e \le \frac{2^{b}}{2}$	$-2^{-b} \le \epsilon \le 2^{-b}$
Truncation	Two's complement	$-2^{-b} < e \le 0$	$-2.2^{-b} < \epsilon \le 0$, $M>0$; $0 \le \epsilon < 2.2^{-b}$, $M<0$
Sign-magnitude One's	"	$-2^{-b} < e \le 0$, $x>0$	$-2.2^{-b} < \epsilon \le 0$

Input/o/p Quantization Error: [Apr. 2018, Nov. 2017]

→ Arises when continuous signal is converted into digital value.

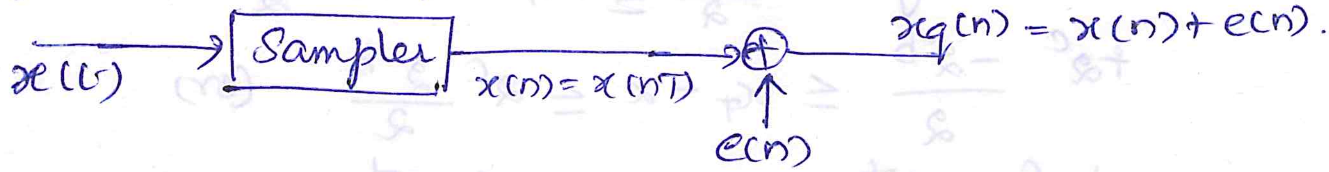
Quantization error $e(n) = x_q(n) - x(n)$.

$x_q(n)$ - sampled quantized value

$x(n)$ - sampled unquantized value.

Quantisation noise:

In digital processing of analog signals, quantization error is referred as additive noise signal (ie) $x_q(n) = x(n) + e(n)$.



Quantisation error $e(n) = x_q(n) - x(n)$.

Variance of $e(n) = \sigma_e^2 = E[e^2(n)] - E^2[e(n)]$.

For rounding, $\sigma_e^2 = \frac{1}{q}\int_{-q/2}^{q/2} e^2(n)\,de - (0)^2 = \frac{q^2}{12}$

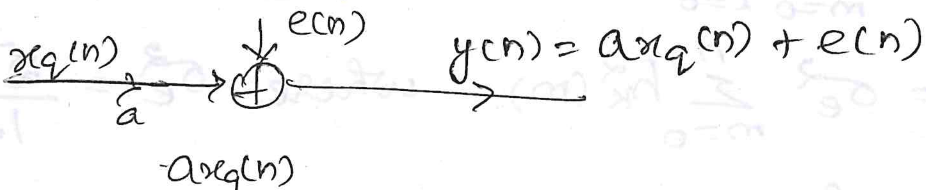
$$\sigma_e^2 = \frac{2^{-2b}}{12}.$$

[Rounding : $-q/2 \le e(n) \le q/2$; mean $= 0$

Truncation : $-q \le e(n) \le 0$; mean $= -q/2$]

Product Quantisation error: [Apr. 2019]

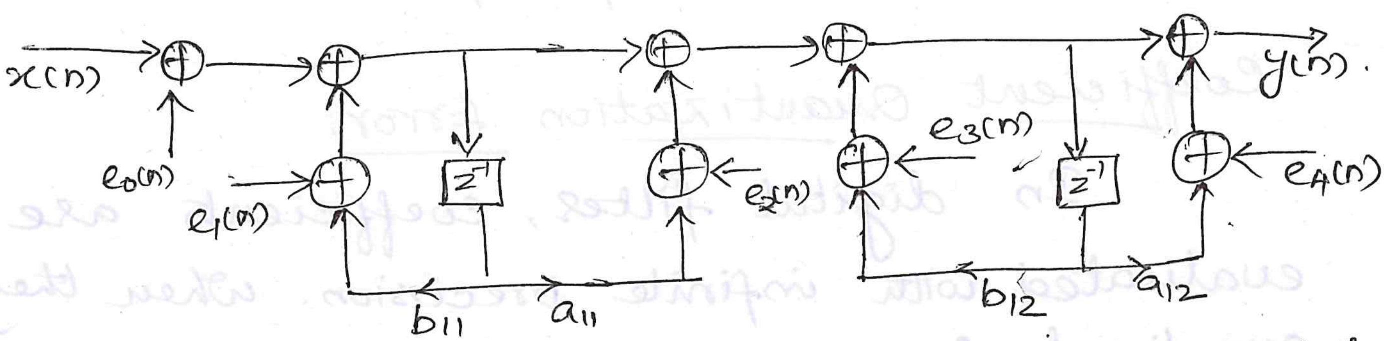
In fixed point arithmetic, product of two b bit numbers result in numbers 2b bits long. In DSP applications, to round this product to a b-bit number, which produce an error known as product quantization error (or) product round off noise.



Assumptions: 1) For any n, error sequence $e(n)$ is uniformly distributed over the range $\frac{-q}{2}$ & $\frac{q}{2}$.
(ie) mean of $e(n)$ is zero. variance $\sigma_e^2 = \frac{2^{-2b}}{12}$

2) Error sequence is stationary white noise seq.

3) Error sequence $e(n)$ is uncorrelated with signal sequence $x(n)$.



Quantization noise model of a cascaded section.

In this model, five noise sources are present. Consider, K^{th} noise source $e_K(n)$. If $h_K(n)$ is filter's impulse response from noise source to filter output, response due to noise source $e_k(n)$ is obtained by convolution as

$$e_k(n) = \sum_{m=0}^{n} h_k(m)\, e_k(n-m).$$

Variance $\sigma_{ek}^2 = E\left[\sum_{m=0}^{n} h_k(m)\, e_k(n-m) \sum_{l=0}^{n} h_k(l)\, e_k(n-l)\right]$

$$= \sum_{m=0}^{n}\sum_{l=0}^{n} h_k(m)\, h_k(l)\, E\left[e_k(n-m)\, e_k(n-l)\right]$$

$$= \sum_{m=0}^{n}\sum_{l=0}^{n} h_k(m)\, h_k(l)\, \underbrace{\delta(l-m)\, \sigma_e^2}_{}$$

proof from white noise stationary proce

$$= \sigma_e^2 \sum_{m=0}^{n} h_k^2(m). \quad \text{where } \sigma_e^2 = \frac{2^{-2b}}{12}.$$

For IIR filter, impulse response approach to zero as $m \to \infty$.

$$\sigma_{ok}^2 = \sigma_e^2 \sum_{m=0}^{\infty} h_k^2(m).$$

Total steady state noise variance $\sigma_o^2 = \sum_k \sigma_{ok}^2$

$$\sigma_{ok}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H_k(z)\, H_k(z^{-1})\, z^{-1}\, dz.$$

$H_k(z) \to$ noise transfer function.

Coefficient Quantization Error:

In digital filter, coefficients are evaluated with infinite precision. When they are quantized, frequency response of actual filter deviates from that which would have been obtained with an infinite word length representation and filter fails to meet desired specifications. If poles of desired filter are close to unit circle, then filter with quantized coefficients may lie outside unit circle leads to instabili

Limit cycle Oscillations due to product quantisation and summation:

Limit cycle is an error which arises due to quantisation.

1) Zero input Limit cycle Oscillations:

When a stable IIR digital filter is excited by finite i/p sequence, o/p will ideally decay to zero. The non linearities due to finite precision arithmetic operations often cause periodic oscillation to occur in the output. Such oscillations are called zero i/p limit cycle oscillations.

The limit cycles occur as a result of quantisation effects in multiplications. The amplitude of o/p during limit cycle are confined to a range of values that is called dead band of filter.

2) Overflow limit cycle oscillations:

In addition to limit cycle oscillations caused by rounding the result of multiplication, there is limit cycle oscillations caused by addition, which make the filter o/p oscillate between maximum and minimum amplitudes. Such limit cycles are overflow oscillations.

Noise Power Spectrum:

In digital processing of analog signals, quantisation error is viewed as additive noise signal, (ie) $x_q(n) = x(n) + e(n)$.

If rounding is used for quantisation, then quantisation error $e(n) = x_q(n) - x(n)$ is bounded by $-\frac{q}{2} \le e(n) \le q/2$.

Variance of $e(n)$ is $\sigma_e^2 = E[e^2(n)] - E^2[e(n)]$

where $E[e^2(n)] \rightarrow$ average of $e^2(n)$.

$E[e(n)] \rightarrow$ mean value of $e(n)$.

$\therefore$ For rounding, $\sigma_e^2 = \frac{1}{q}\int_{-q/2}^{q/2} e^2(n)\,de - (0)^2 = \frac{q^2}{12}$.

$$\sigma_e^2 = \frac{(2^{-b})^2}{12} = \frac{2^{-2b}}{12}$$

$\frac{1}{q}\cdot\frac{e^3(n)}{3}\Big]_{-q/2}^{q/2} = \frac{1}{3q}\left(\frac{q^3}{8}+\frac{q^3}{8}\right) = \frac{q^2}{3\cdot 4}$

In two's complement truncation, $e(n)$ lies between 0 and $-q$ having mean value of $\frac{-q}{2}$.

Variance or power of error signal $e(n)$ is

$$\sigma_e^2 = \frac{1}{q}\int_{-q}^{0} e^2(n)\,de - \left[\frac{-q}{2}\right]^2$$

$$= q^2/3 - q^2/4 = q^2/12$$

In both cases, $\sigma_e^2 = \frac{2^{-2b}}{12}$, which is known as steady state noise power due to i/p quantisation

If i/p signal is $x(n)$ & its variance is σ_x^2, then ratio of signal to noise power that is signal to noise ratio for rounding is

$$\frac{\sigma_x^2}{\sigma_e^2} = \frac{\sigma_x^2}{2^{-2b}/12} = 12(2^{2b}\sigma_x^2).$$

When expressed in a log scale S/N ratio in dB

$$= 10\log_{10}\frac{\sigma_x^2}{\sigma_e^2} = 6.02b + 10.79 + 10\log_{10}\sigma_x^2.$$

SNR increases approximately 6 dB for each bit added to register length.

If i/p signal is $Ax(n)$ instead of $x(n)$ where $0 < A < 1$, then the variance is $A^2\sigma_x^2$.

Hence, $SNR = 10\log_{10}\frac{\sigma_x^2 A^2}{\sigma_e^2} = 6b + 10.8 + 10\log_{10}\sigma_x^2 + 20\log_{10}A.$

if $A = \frac{1}{4\sigma_x}$, $SNR = 6b - 1.24$ dB.

Thus, to obtain $SNR \geq 80$ db requires $b = 14$ bits.

O/P Noise Power:

Let $\epsilon(n)$ be the o/p noise due to quantisation of the i/p. We get, $\epsilon(n) = e(n) * h(n)$.

$$= \sum_{k=0}^{n} h(k)\, e(n-k)$$

The variance of any term in the above sum is equal to $\sigma_e^2 h^2(n)$.

Variance of sum of independent random variable is the sum of their variances.

If the quantisation errors are assumed to be independent at different sampling instances, then the variance of o/p,

$$\sigma_e^2(n) = \sigma_e^2 \sum_{n=0}^{K} h^2(n).$$

To find the steady state variance, extend the limit K upto infinity.

Then, $\sigma_e^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n).$

Using Parseval's theorem, steady state o/p noise variance due to quantisation error is given by

$$\sigma_e^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) = \frac{\sigma_e^2}{2\pi j} \oint_c H(z)\, H(z^{-1})\, z^{-1}\, dz.$$

where the closed contour of integration is around the unit circle $|z| = 1$ in which case only the poles that lie inside the unit circle are evaluated using the residue theorem.

Limit cycle Oscillations:

1) Zero input limit cycle oscillations:

When a stable IIR filter is excited by a finite input sequence (ie) constant, the output will ideally decay to zero. The nonlinearities due to the finite precision arithmetic operations cause periodic oscillations to occur in the o/p. Such oscillations in recursive systems are called zero input limit cycle oscillations.

Consider first order IIR filter with difference equation $y(n) = x(n) + \alpha\, y(n-1)$.

Let assume $\alpha = 1/2$, data register length 3bits + sign bit.

If i/p $x(n) = \begin{cases} 0.875 & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases}$

n	$x(n)$	$y(n-1)$	$\alpha y(n-1)$	$Q[\alpha y(n-1)]$	$y(n) = x(n) + Q[\alpha y(n-1)]$
0	0.875	0.0	0.0	0.000	7/8
1	0	7/8	7/16	0.100	1/2
2	0	1/2	1/4	0.010	1/4
3	0	1/4	1/8	0.001	1/8
4	0	1/8	1/16	0.001	1/8
5	0	1/8	1/16	0.001	1/8

Q[.] represents bounded operation.

For $n \geq 3$, o/p remains constant and gives 1/8 as steady o/p causing limit cycle behaviour.

Dead band: The limit cycles occur as a result of quantization effects in multiplications. The amplitudes of o/p during limit cycle are confined to a range of values that is dead band.

Consider single pole IIR system whose difference equation is given by

$$y(n) = \alpha y(n-1) + x(n) \quad , \quad n > 0$$

After rounding the product term,

$$y_q(n) = Q[\alpha y(n-1)] + x(n)$$

During limit cycle oscillations,

$$Q[\alpha y(n-1)] = y(n-1) \quad \text{for } \alpha > 0$$
$$= -y(n-1) \quad \text{for } \alpha < 0$$

By rounding, $|Q[\alpha y(n-1)] - \alpha y(n-1)| \leq \frac{2^{-b}}{2}$

$$|y(n-1)| - |\alpha y(n-1)| \leq \frac{2^{-b}}{2}$$

$$y(n-1) \leq \frac{\frac{1}{2} 2^{-b}}{1-|\alpha|} \longrightarrow \text{deadband}$$

Overflow limit cycle Oscillations (overflow error)

Based on addition, filter o/p oscillates between maximum and minimum amplitudes. Such limit cycles is referred as overflow oscillations.

Consider two positive numbers n_1 & n_2

$n_1 = 0.111 \longrightarrow 7/8$

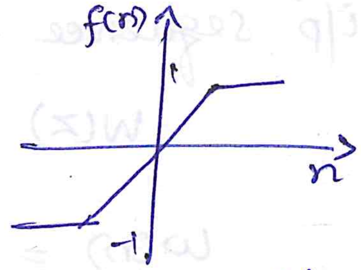
$n_2 = 0.110 \longrightarrow 6/8$

$n_1 + n_2 = 1.101 \longrightarrow -5/8$ in sign magnitude.

In the above example, two +ve no.s are added, the sum is wrongly interpreted as negative number.

Here, n – i/p of adder
$f(n)$ – o/p of adder.

When overflow is detected, sum of adder is set equal to max. value.



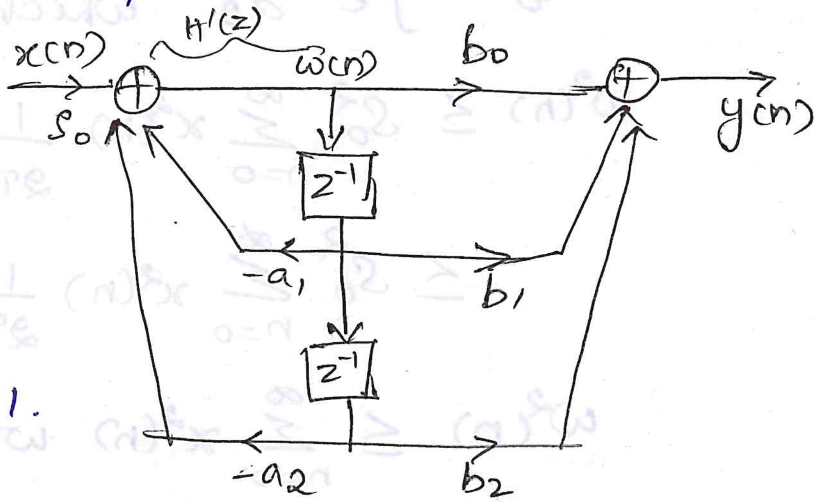
Saturation adder transfer characteristics

Signal Scaling to Prevent overflow:

[Apr. 2018, 2019]

Saturation arithmetic eliminates limit cycles due to overflow, but it causes signal distortion due to non linearity of the clipper. In order to limit the amount of non-linear distortion, it is important to scale the i/p signal and unit sample response between the i/p and any internal summing node in the system such that overflow becomes a rare event.

In fig, scale factor S_0 is introduced between i/p $x(n)$ & adder 1, to prevent overflow at o/p adder 1.



Second order IIR filter.

Overall I/o transfer function

$$H(z) = S_0 \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = S_0 \frac{N(z)}{D(z)}$$

$$H'(z) = \frac{W(z)}{X(z)} = \frac{S_0}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{S_0}{D(z)}$$

If the instantaneous energy in o/p sequence $w(n)$ is less than finite energy in the i/p sequence then, there will not be any overflow.

$$W(z) = \frac{S_0 X(z)}{D(z)} = S_0 S(z) X(z), \quad S(z) = \frac{1}{D(z)}$$

$$w(n) = \frac{S_0}{2\pi} \int S(e^{j\theta}) X(e^{j\theta}) e^{jn\theta} d\theta$$

which gives $w^2(n) = \frac{S_0^2}{4\pi^2} \left| \int S(e^{j\theta}) X(e^{j\theta}) e^{jn\theta} d\theta \right|^2$

Using Schwartz inequality,

$$w^2(n) \leq S_0^2 \left[\frac{1}{2\pi} \int_{2\pi} |S(e^{j\theta})|^2 d\theta \right] \left[\frac{1}{2\pi} \int_{2\pi} |X(e^{j\theta})|^2 d\theta \right]$$

Apply parseval's theorem, we get

$$w^2(n) \leq S_0^2 \sum_{n=0}^{\infty} x^2(n) \frac{1}{2\pi} \int_{2\pi} |S(e^{j\theta})|^2 d\theta$$

Let $z = e^{j\theta}$. Differentiate w.r.t. θ, we have

$dz = j e^{j\theta} d\theta$ which gives $d\theta = \frac{dz}{jz}$

$$w^2(n) \leq S_0^2 \sum_{n=0}^{\infty} x^2(n) \frac{1}{2\pi j} \oint_c |S(z)|^2 z^{-1} dz$$

$$\leq S_0^2 \sum_{n=0}^{\infty} x^2(n) \frac{1}{2\pi j} \oint_c S(z) S(z^{-1}) z^{-1} dz$$

$$w^2(n) \leq \sum_{n=0}^{\infty} x^2(n) \text{ when } S_0^2 \frac{1}{2\pi j} \oint_c S(z) S(z^{-1}) dz = 1$$

which gives us $S_0^2 = \dfrac{1}{\frac{1}{2\pi j} \oint_c S(z) S(z^{-1}) z^{-1} dz}$

$$= \frac{1}{\frac{1}{2\pi j} \oint_c \frac{z^{-1} dz}{D(z) D(z^{-1})}} = \frac{1}{I}; \quad I = \frac{1}{2\pi j} \oint_c \frac{z^{-1} dz}{D(z) D(z^{-1})}$$

1) Find the steady state variance of the noise in the o/p due to quantization of i/p for the first order filter. $y(n) = a\,y(n-1) + x(n)$.

Solution: i) Impulse response of filter is $h(n) = a^n u(n)$.

$$\sigma_E^2 = \sigma_\epsilon^2 \sum_{k=0}^{\infty} h^2(n) = \sigma_\epsilon^2 \sum_{k=0}^{\infty} a^{2n} = \sigma_\epsilon^2 [1 + a^2 + a^4 + \dots \infty]$$

$$= \sigma_\epsilon^2 \frac{1}{1-a^2} = \frac{2^{-2b}}{12}\left[\frac{1}{1-a^2}\right]$$

(or)

method ii) $y(n) = a\,y(n-1) + x(n)$.

Take z transform,

$$Y(z) = a z^{-1} Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}.$$

$$H(z^{-1}) = \frac{z^{-1}}{z^{-1}-a}$$

$$\sigma_E^2 = \sigma_\epsilon^2 \frac{1}{2\pi j} \oint_c H(z)\, H(z^{-1})\, z^{-1}\, dz.$$

$$= \sigma_\epsilon^2 \frac{1}{2\pi j} \oint \frac{z}{z-a} \frac{z^{-1}}{z^{-1}-a} z^{-1}\, dz.$$

$$= \sigma_\epsilon^2 \frac{1}{2\pi j} \oint_c \frac{z^{-1}}{(z-a)(z^{-1}-a)} dz$$

$$= \sigma_\epsilon^2 \Big[\text{residue of } \frac{z^{-1}}{(z-a)(z^{-1}-a)} \text{ at } z=a \; +$$

$$\underbrace{\text{residue of } \frac{z^{-1}}{(z-a)(z^{-1}-a)} \text{ at } z = 1/a}_{\text{equal zero}}\Big]$$

$$= \sigma_\epsilon^2 \left[(z-a) \frac{z^{-1}}{(z-a)(z^{-1}-a)} \Big|_{z=a} \right] = \sigma_\epsilon^2 \frac{a^{-1}}{a^{-1}-a} = \sigma_\epsilon^2 \frac{1}{1-a^2}.$$

2) The o/p of an ADC is applied to a digital filter with the system function $H(z) = \frac{0.5z}{z-0.5}$. Find the o/p noise power from the digital filter, when the i/p signal is quantised to have 8 bit.

Solution: Quantisation noise power is

$$\sigma_e^2 = \frac{2^{-2b}}{12} = \frac{2^{-16}}{12} = 1.27 \times 10^{-6}.$$

o/p noise power is given by,

$$\sigma_{eo}^2 = \frac{\sigma_e^2}{2\pi j} \oint_c H(z)\, H(z^{-1})\, z^{-1}\, dz$$

$$= \frac{\sigma_e^2}{2\pi j} \oint_c \left(\frac{0.5z}{z-0.5}\right)\left(\frac{0.5z^{-1}}{z^{-1}-0.5}\right) z^{-1}\, dz$$

$$= \frac{\sigma_e^2}{2\pi j} \oint_c \frac{0.25}{(z-0.5)(1-0.5z)}\, dz$$

$\sigma_{eo}^2 = \sigma_e^2 I$ where $I = \frac{1}{2\pi j} \oint_c \frac{0.25}{(z-0.5)(1-0.5z)}\, dz$

By residue method, integrate it.

I = Sum of residues at the poles within unit circle (i.e.) within $|z|<1$. The poles are at $z=0.5$ and $z=2$.

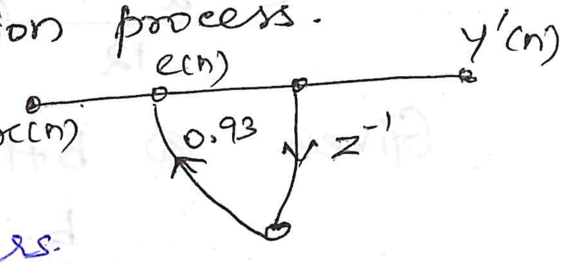
I = residue at $z=0.5$

$$I = \cancel{(z-0.5)} \frac{0.25}{\cancel{(z-0.5)}(1-0.5z)}\Bigg|_{z=0.5}$$

$$= \frac{0.25}{1-(0.5)(0.5)} = \frac{0.25}{0.75} = 0.333$$

$$\sigma_{eo}^2 = 1.27 \times 10^{-6} \times (0.333) = 0.423 \times 10^{-6}.$$

3) For the recursive filter shown below the o/p $x(n)$ has a peak value of 10V, represented by 6 bits. Compute the variance of o/p due to A/D conversion process.



Solution:

Assume 2's complement representation for binary numbers.

Quantization step size $q = \frac{R}{2^b}$

$R = 10$ & $b = 6$. $\quad q = \frac{10}{2^6} = 0.15625$.

Variance of error signal $\sigma_e^2 = \frac{q^2}{12}$.

$$= \frac{0.15625^2}{12} = 2.0345 \times 10^{-3}.$$

4) The i/p to the system $y(n) = 0.999y(n-1) + x(n)$ is applied to ADC. What is the power produced by the quantisation noise at the o/p of filter if the i/p is quantized to a) 8 bits b) 16 bits.

Solution: $\quad y(n) = 0.999\, y(n-1) + x(n)$.

Take z-transform

$$Y(z) = 0.999\, z^{-1} Y(z) + X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.999 z^{-1}}$$

Inv. z transform

$$h(n) = (0.999)^n u(n).$$

Quantization noise power at the o/p of digital filter is $\sigma_{eo}^2 = \sigma_e^2 \sum_{k=0}^{\infty} h^2(k) = \sigma_e^2 \sum_{k=0}^{\infty} (0.999)^{2k}$

$$= \sigma_e^2 \frac{1}{(1-0.999)^2} = \sigma_e^2 (500.25)$$

$$= \frac{2^{-2b}}{12} (500.25)$$

Given ⓐ $b+1 = 8$ bits (Assume including sign bit)

$b = 7$

$$\sigma_{eo}^2 = \frac{2^{-14}}{12} (500.25) = 2.544 \times 10^{-3}.$$

ⓑ $b+1 = 16$ bits

$b = 15$

$$\sigma_{eo}^2 = \frac{2^{-30}}{12} (500.25) = 3.882 \times 10^{-8}.$$

5) For second order IIR filter

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

. Study the effect of shift in pole location with 3 bit coefficient representation in direct form.

Soln: $$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

$$= \frac{1}{z^{-1}(z-0.5)(z-0.45)z^{-1}}$$

$$H(z) = \frac{z^2}{(z-0.5)(z-0.45)}.$$

The original poles of $H(z)$ be P_1 & P_2. $\boxed{P_1 = 0.5}$ $\boxed{P_2 = 0.45}$

1) Direct form

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

$$H(z) = \frac{1}{1-0.5z^{-1}-0.45z^{-1}+0.225z^{-2}}$$

$$= \frac{1}{(1-0.95z^{-1}+0.225z^{-2})}$$

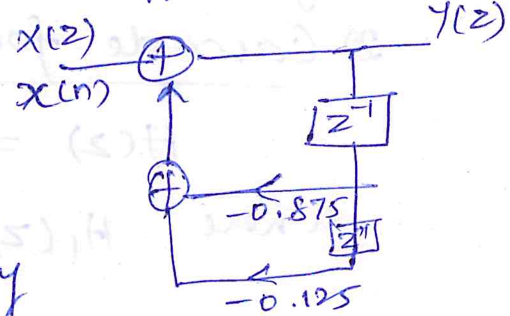
Let us quantize the coefficients by truncation

$$.95 \longrightarrow .1111_2 \xrightarrow[3\text{ bits}]{\text{Truncate}} .111_2 \longrightarrow .875_{10}$$

$$.225 \longrightarrow .001l_2 \longrightarrow .001_2 \longrightarrow .125_{10}$$

Let $\bar{H}(z)$ be the transfer function of IIR system after quantizing the coefficients.

$$\bar{H}(z) = \frac{1}{1-.875z^{-1}+0.125z^{-2}}$$



The poles of $\bar{H}(z)$ are given by truncated roots of the denominator polynomial $\bar{H}(z)$. Let the poles of $\bar{H}(z)$ be $\bar{P}_{d1}$, $\bar{P}_{d2}$.

The roots of quadractic are given as

$$z^2 - .875z + .125 = 0.$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-0.875 \pm \sqrt{0.875^2-4\times 0.175}}{2}$$

$$= 0.695 \ \& \ 0.18$$

$0.695_{10} \longrightarrow .11011_2 \longrightarrow .101_2 \longrightarrow .625_{10}$

$0.18_{10} \longrightarrow 0.00101_2 \longrightarrow .001_2 \longrightarrow .125_{10}$

$P_{d_1} = 0.625$ & $P_{d_2} = 0.125$

By comparing the poles of $H(z)$ & $\bar{H}(z)$, we observe that both quantized and unquantized poles of $\bar{H}(z)$ deviate verymuch from original poles.

2) Cascade form:

$$H(z) = H_1(z) \cdot H_2(z)$$

where $H_1(z) = \dfrac{1}{1-0.5z^{-1}}$, $H_2(z) = \dfrac{1}{1-0.45z^{-1}}$

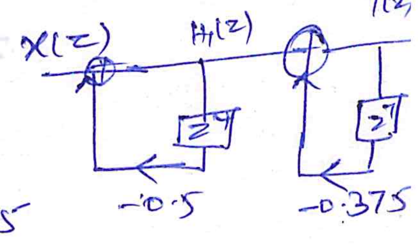
By quantizing the coefficients of $H_1(z)$ & $H_2(z)$ by truncation.

$.5_{10} \longrightarrow .1000_2 \longrightarrow .100_2 \longrightarrow .5_{10}$

$.45_{10} \longrightarrow .0111_2 \longrightarrow .011_2 \longrightarrow .375_{10}$

After quantizing $\bar{H}_1(z) = \dfrac{1}{1-0.5z^{-1}}$

$$\bar{H}_2(z) = \frac{1}{1-0.375z^{-1}}$$



Quantized poles of system $P_{e_1} = 0.5$
$P_{e_2} = 0.375$

Comparing this poles with original one, we observe that one of the pole is same & other pole is very close to original pole.

UNIT V INTRODUCTION TO DIGITAL SIGNAL PROCESSORS

The programmable digital signal processors are general purpose microprocessors designed for dsp applications. They contain special architecture and instruction set to execute computation.

DSP functionalities:

The goal of DSP is to measure, filter or compress continuous real world analog signals. DSPs have better power efficiency, they are more suitable in portable devices like mobile phones because of power consumption constraints.

DSP algorithms require a large no. of mathematical operations to be performed quickly and repeatedly on a series of data samples.

Circular buffering:

A circular buffer is a memory allocation scheme where memory is reused when an index, incremented modulo the buffer size, writes over a previously used location.

A circular buffer makes a bounded queue when seperate indices are used for inserting and removing data.

Circular buffering is an efficient method of storing i/p data of a real time system. A DSP processor uses dedicated hardware to provide fast circular buffers.

To implement FIR filter, calculate o/p samples from i/p, $x(n)$, $x(n-1)$ ... $x(n-7)$. These are stored in memory & updated.

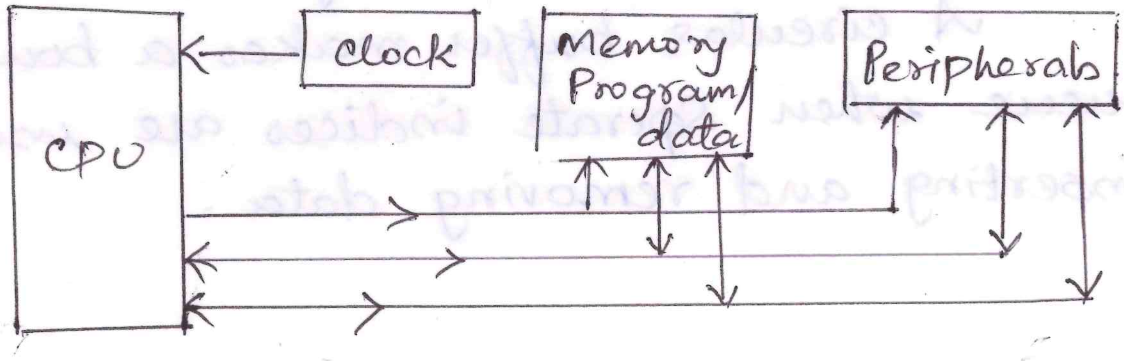
First, place the samples in consecutive locations.
The end of array is connected to begining → circular buffer.
When new sample is acquired, it replaces old one & pointer is moved to one address ahead.

At instant: Samples in buffer	memory address	After next: Samples
$x(n-3) \rightarrow$	2041	$\leftarrow x(n-4)$
$x(n-2) \rightarrow$	2042	$\leftarrow x(n-3)$
$x(n-1) \rightarrow$	2043	$\leftarrow x(n-2)$
$x(n) \rightarrow$	2044	$\leftarrow x(n-1)$
$x(n-7) \rightarrow$	2045	$\leftarrow x(n)$
$x(n-6) \rightarrow$	2046	$\leftarrow x(n-7)$
$x(n-5) \rightarrow$	2047	$\leftarrow x(n-6)$
$x(n-4) \rightarrow$	2048	$\leftarrow x(n-5)$

Circular buffers are efficient because only one value changed when new sample is acquired.
(ie) In new value, $x(n)$ only changed in loc. 2045

DSP architecture:

Von Neumann Architecture:



In this, program instructions were stored in ROM. CPU reads/writes data from/to the memory. Both can't occur at same time, since instruction and data use the same signal pathways and memory. It consists of 3 buses:

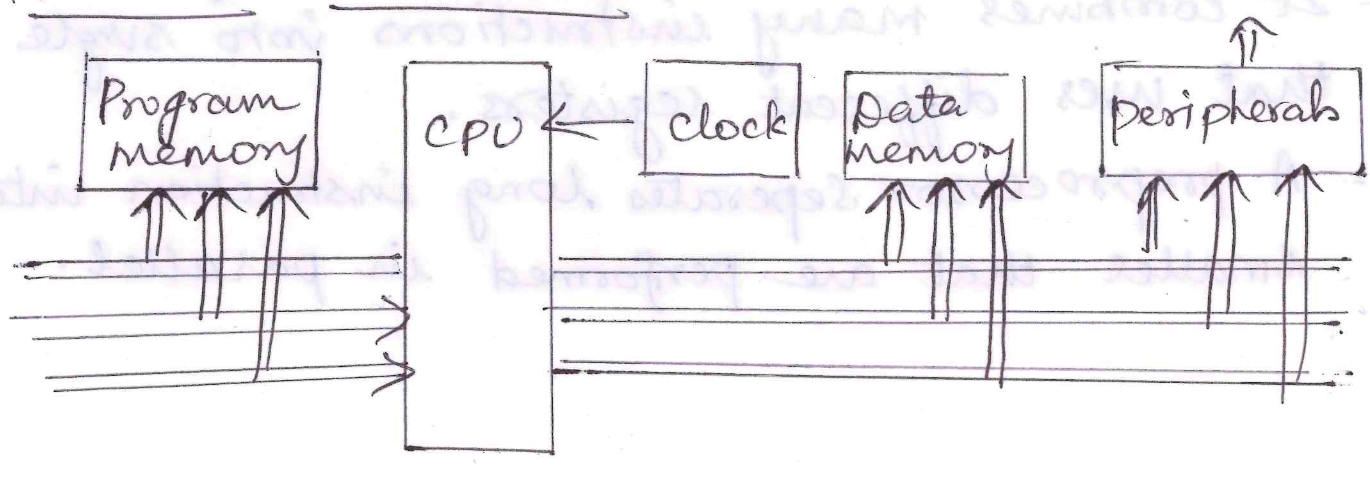
Data bus: → Transports data b/w CPU & peripherals
→ bidirectional.
→ CPU reads/writes data in peripherals

Address bus: CPU uses this, to indicate which peripherals it wants to access and within each peripheral which specific register. CPU writes address which is read by peripherals. - Unidirectional

Control bus: Bus carries signals used to manage and synchronize the exchanges between CPU & its peripherals, & indicate if CPU wants to read/write the peripheral.

Main characteristic is that it only possesses 1 bus system. The same bus carries all information exchanged between CPU & peripheral.

Harvard Architecture:



Harrard architecture seperates memories for instructions and data. So instructions & operands canbe fetched simultaneously.

Since, it has two memories, it is not possible for CPU to mistakenly write codes into program memory & compute the code while it is executing.

It is less flexible.

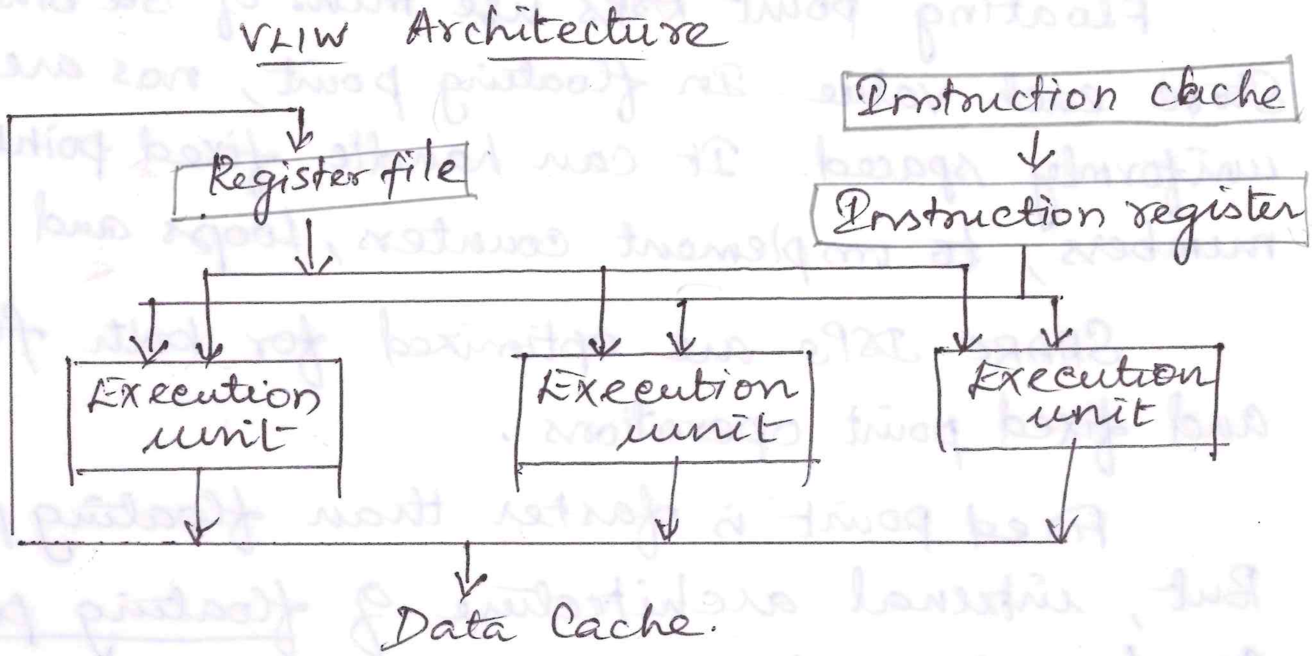
Modified Harrard architecture used DSPs multiport memory and i/o peripherals. It has multiple bus system for pgm memory/data memory alone. It increases complexity, but allows to access several memory locations simultaneously, by increasing data thro'put between memory & CPU.

VLIW Architecture:

VLIW - Very Long Instruction Word.

- It increases no. of instructions/cycle. It is concatenation of short instructions & require multiple units for execution, running in parallel.
- It combines many instructions into single that uses different registers.
- A preprocessor seperates long instruction into smaller that are performed in parallel.

VLIW Architecture



Advantages:
Increased performance, Better compiler targets
Easier to program, Potentially capable.

Disadvantages:
Increased memory use, high power consumption
Program must keep track of instruction Scheduling

Fixed and Floating point architecture principles

Fixed point DSPs represent each number with a min. of 16 bits. Four common ways to represent a number. In unsigned integer, stored number takes integer value from 0 to 65,535. Signed integer uses two's complement to make range include negative numbers, from -32,768 to 32,767.

Unsigned fraction, 65,536 levels spread between 0 & 1
Signed fraction spaced between -1 & 1.

Floating point DSPs use min. of 32 bits to Store each value. In floating point, nos are not uniformly spaced. It can handle fixed point numbers, to implement counters, loops and signals

SHARC DSPs are optimized for both floating and fixed point operations.

Fixed point is faster than floating point DSPs But, internal architecture of floating point is Complex. All registers are 32 bit wide, multiplier ALU quickly perform arithmatic. Large instruction set. Better precision, high dynamic range

Fixed point DSP is cheaper than floating point DSP. It is also faster.

Programming:

Ex.1: Extended Precision Addition:

$$\begin{array}{cccc} X_3 & X_2 & X_1 & X_0 \\ Y_3 & Y_2 & Y_1 & Y_0 \\ \hline W_3 & W_2 & W_1 & W_0 \end{array}$$

```
LDP  #100H        ; ACC = X1 00
LACC 0001,10        = X1 X0
ADDS 0000           = X1 X0 + 00 Y0
ADDS 0004           = X1 X0 + Y1 Y0
ADD  0005,10H     ACCL = W0
SACL 0008         ACCH = W1
SACH 0009         Opern upto 2nd 32 bits
LACC 0003,10      ACC = X3 00
ADDC 0002           = X3 X2 + carry
ADDS 0006           = X3 X2 + 00 Y2 + Carry
ADD  0007,10H       = X3X2 + Y3Y2 + Carry
SACL 0010         ACCL = W2 (result).
SACH 0011         ACCH = W3
```

Ex. 2 : Extended precision subtraction:

```
LDP   #100H
LACC  0001,10   ; ACC = X1 00
ADDS  0000      = X1 X0
SUBS  0004      = X1 X0 + 00 Y0
SUB   0005,10   = X1 X0 - Y1 Y0
SACL  0008      ACCL = W0
SACH  0009      ACCH = W1
LACC  0002,0    ACC = 00 X2
SUBB  0006      ACC = 00X2 - 00Y2 - C
ADD   0003,10   ACC = X3X2 - 00Y2 - CARRY.
SUB   0007,10   = "  - Y3Y2 - "
SACL  000A,0    ACCL = W2
SACH  000B,0    ACCH = W3.
B.
```

Ex. 3 : Integer Multiplication.

```
LDP   #100H
LACC  #037AH,0
SACL  0000
LACC  #012EH,0
SACL  0001,0
LT    0000
MPY   001
PAC
SACL  0002,0
SACH  0003,0
H :   B
END
```

Ex. 4: Two's complement of a given number:

```
LDP   #100H
LACL  #5
CMPL
ADD   #1
SACL  0000,0
H:    B
```

Applications:

1) Communication Systems: DSPs applied to implement various communication systems.

Ex: Caller ID, cordless handset.

- In voice communication, acoustic-echo canceller based for hands-free wireless system.
- A telephone voice dialer is implemented with 16-bit DSP.
- Modern DSPs for error correction in digital comm
- System prototyping is by DSP due to low cost & easy programming.
- Navigation using GPS is accepted for commercial applⁿ like electronic direction finding.
- TMS320C30 performs correlation, FFT, digital filtering, decimation and demodulation.
- PDSP front-end performs pulse compression, moving target indication (MTI) and constant false alarm (CFA) rate detection.

2) Audio Signal Processing: PDSP applications to audio signal processing canbe classified into 3 categories based on qualities, audible range of signal professional audio products & consumer audio products.

3) Control and data acquisition:

Motorola PDSPs function as powerful micro controllers. Its 56-bit accumulator provides 8 bit extension registers with saturation arithmetic. The o/p noise power due to round off noise of 24-bit is 65,536 times less than 16 bit PDSP.

4) Bio metric information Processing:

In bio metrics, handwritten signature verification, one of the biometric authentication techniques is cheap, realiable and non-intrusive to the person being authorized. This verification method can be part of variety of entrance monitoring and security systems

5) Image / Video Processing:

JPEG 2000 is based on DWT. JPEG, MPEG are implemented in modern digital cameras and digital camcorders.

In medical imaging, DSP is used as on-line data processor for processing MRI.

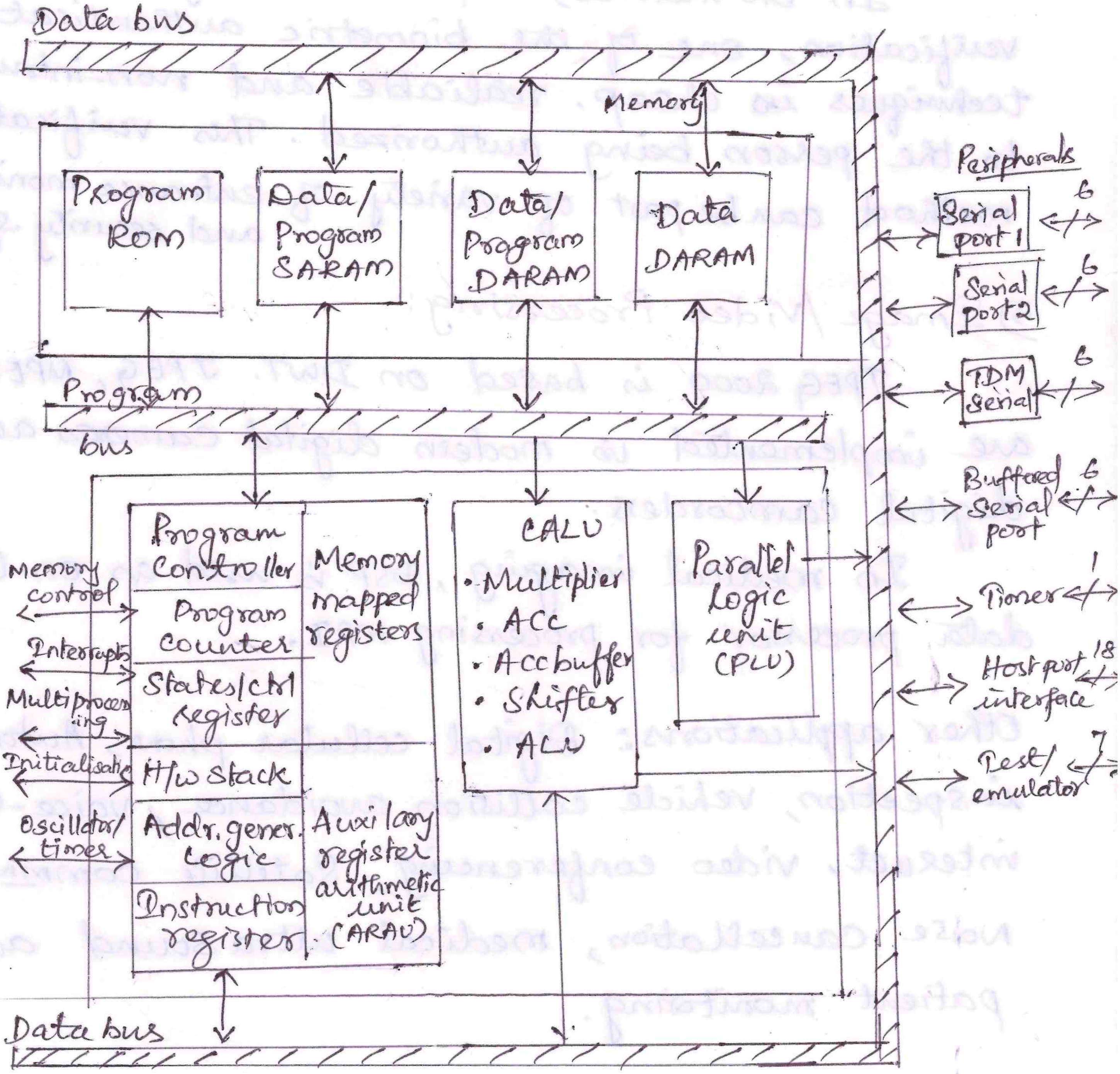
Other applications: Digital cellular phone, Automated inspection, vehicle collision avoidance, voice-over-internet, video conferencing, Satellite commn, Noise cancellation, medical ultra sound and patient monitoring.

Architecture of TMS320C50:

- Fabricated with CMOS technology.
- Fixed point, 16 bit processor at 40MHz.
- Single instruction execution time 50ns.

Functional block diagram of TMS320C50:

Blocks: 1) Bus structure 2) Central Processing unit 3) Onchip memory 4) On chip peripherals.



Bus Structure:

The 'C5X' architecture has four buses:

1) Program bus (PB) 2) Program Address Bus (PAB)
3) Data read bus (DB) 4) Data read address bus (DAB)

PB → carries instruction code & immediate operands from pgm memory to CPU

PAB → provides address to program memory space for both R/W.

DB → interconnects various elements of CPU to data memory space.

DAB → provides address to access data memory space

Central Processing unit:

Elements: i) Central Arithmetic Logic Unit (CALU)
ii) Parallel Logic unit (PLU)
iii) Auxilary Register Arithmetic Unit (ARAU)
iv) Memory mapped registers
v) Program controller.

CALU: CPU uses CALU to perform 2's complement arithmetic. It consists of 16 x16 bit parallel multiplier, 32-bit acc, & acc. buffer, product register, additional shifter.

PLU: Executes logic operations on data without affecting the contents of ACC. It can set, clear, test or toggle multiplier bit in a status/control register

ARAU: 'C5X' consists of register file containing 8 aux. registers (AR0-AR7) each of 16 bit length, 3 bit aux. register pointer (ARP), unsigned 16 bit ALU. Aux. registers file is connected to ARAU.

Index Register (INDX):

16 bit INDX - used by ARAU as a step value to modify the address in AR.

- added to or subtracted from current AR
- used to increment/decrement address in steps larger than 1.

Auxiliary Register Compare Register (ARCR):

16 bit ARCR - used for address boundary comparison. It limits block of data and supports logical comparison.

Block move address register (BMAR):

- holds an address value of source destination space of a block move - hold address of an operand in program memory for a multiply accumulate operation.

$AR_0 - AR_7$:

- can be accessed by CALU & modified by ARAU or PLU - provide 16 bit address for indirect addressing to data space.

Instruction Register (IREG):

- hold the opcode of instruction being executed.

Interrupt Register (IMR, IFR):

- IMR masks special interrupts at required time. - IFR (flag) - indicates the current status of interrupts.

Memory mapped Registers:

'C5x' has 96 registers mapped into page 0 of memory space which contains control/status registers include CPU, serial port, timer & S/w-wait generator.

Program Controller:

- contains logic ckts that decodes the operational instructions, manages CPU pipeline, stores the status of CPU operations and decodes conditional operations.

Elements:

1) Program counter
2) Status and Control registers
3) Hardware stack
4) Address generation logic
5) Instruction register.

Program Counter:

- Contains the address of internal/external program memory used to fetch instructions.

Hardware Stack:

- Stack is 16 bit wide & 8 levels deep & is accessible via PUSH & POP instructions.

Program memory address generation:

- Contains code for application and holds table information and immediate operands. Pgm memory is accessed by program address.

Status and control Register:

Four of these registers - circular buffer control register, process mode status register, status registers ST0, ST1.

Circular buffer Registers:

CBSR1, CBSR2 → 16 bit registers that hold address when the circular buffer starts.

CBER1, CBER2 → indicate address when circular buffer ends.

CBCR → controls the operation of these registers.

On-chip memory:

Total address range 224 K words x 16 bits.

Segments: 64 k word - program memory space, local data memory space, I/O ports.

32 k word - Global data memory space.

Large on chip memory 'C5x' includes

Program ROM

Data/program single access RAM (SARAM)

Data/program Dual Access RAM (DARAM)

Program ROM - 16 bit on-chip maskable PROM.

DARAM - 512 word data/program DARAM block B0.
512 word data DARAM block B1,
32 word data DARAM block B2.

SARAM - 2k word & 1K word block.

Config. - data memory, pgm memory, both data/program memory.

Instruction Cache Memory:

- Cache memory is used with instruction register - Used to store previous 16 executed instruct. Instruction fetched for instruction register is stored in instruction cache. After executing current instruction, instruction cache feeds next instruction to register. This increases speed of operation.

Cache memory monitor keeps track of program memory address of instruction. Stored Cache memory maintains track of valid instruction that are before & after currently executed instruction.

TMS 320C54x Processors:

- Advanced modified Harvard Architecture
- Contains all features of basic & additional features for improved speed & performance
- Upward compatible to earlier fixed point processors - C2x, C2xx & C5x processors.
- 16 bit fixed point DSP family.

Advantages:

1. Enhanced Harvard architecture, which includes one program bus, 3 databuses & 4 address buses
2. CPU has high degree of parallelism & application specific h/w logic.
3. highly specialized instruction set for faster algorithms.
4. Increased performance & low power consumption.

Features of 'C54X':

A) CPU: 1. One program bus, 3 data buses, 4 address buses
2) 40 bit ALU, includes parallel shifter & two independent 40 bit accumulators.
3) 17 x 17 bit 11^l multiplier coupled to 40 bit adder for non pipelined single cycle MAC operation.
4) Compare, Select, Store Unit (CSSU) for add/compare selection of viterbi operator.
5) Exponent encoder to compute exponent of 40 bit accumulator in single cycle.
6) Two address generators, including 8 auxillary registers & 2 aux. register arithmetic units.
7) Multiple CPU/core architecture on some devices.

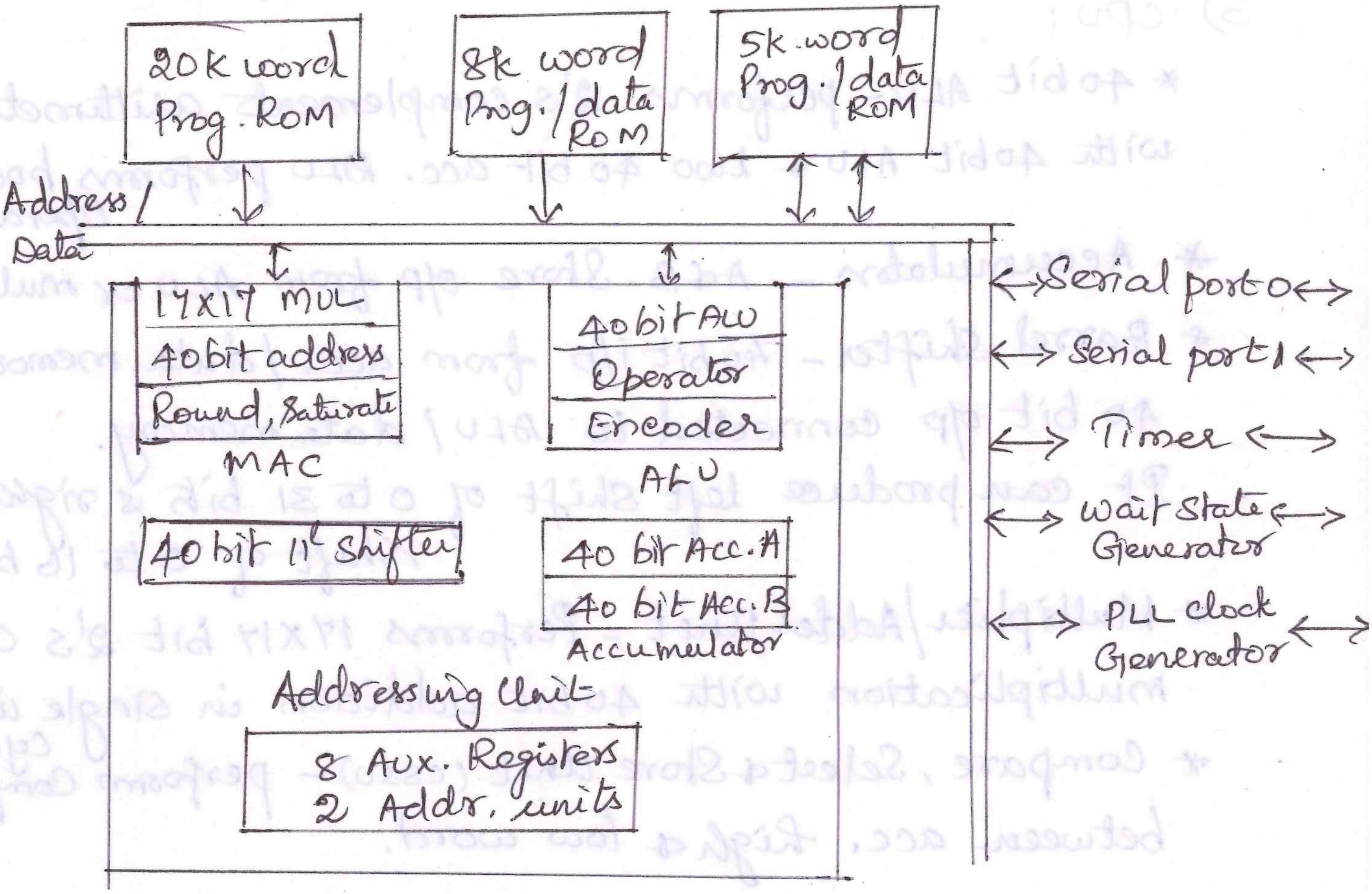
B) Memory: 1) 192k words x 16 bit addressable memory space
2) Extended programmable memory in some devices

C) Instruction Set:
1) Single instruction repeat & block repeat operations
2) Block memory move operations
3) 32 bit long operand instructions
4) 11^l load & store instructions.
5) Conditional store instructions
6) Fast return from interrupt.

D) Onchip Peripherals:

1) S/w programmable wait state generator.
2) Programmable bank switching logic.
3) On chip PLL generator.
4) External bus off control.
5) Programmable timer.
6) Bus hold feature for data bus.

C54x Architecture:



1) Bus: * 8 major 16 bit buses (4 prog./data bus & 4 address bus)
* Program bus carries instruction code & immediate operands from program memory.
* 3 address bus interconnect CPU, data address generation logic, Prog. address generation logic, on chip peripherals & data memory.

* 4 address buses carry address needed for instruction execution.

2) Internal Memory Organisation:

* 3 individually selectable spaces: prog, data & I/O space.
* 26 CPU registers, peripheral registers that are mapped in data memory space.
* Contains RAM & ROM.
* On chip ROM is part of prog. memory space & somecases part of data memory space.

3) CPU:

* 40bit ALU - performs 2's complement arithmetic with 40bit ALU & two 40 bit acc. ALU performs boolean operations.
* Accumulators - A&B. Store o/p from ALU or multiplier
* Barrel Shifter - 40bit i/p from acc./data memory. 40 bit o/p connected to ALU/data memory. It can produce left shift of 0 to 31 bits & right shift of 0 to 16 bits
* Multiplier/Adder unit - Performs 17X17 bit 2's comp. multiplication with 40bit addition in single instrns cycle.
* Compare, Select & Store unit (CSSU) - performs comparisons between acc. high & low word.

Data addressing modes:

1. Immediate 2. Register 3. Direct 4. Indirect
5. Memory mapped register 6. Circular addressing

Program memory addressing:

It is addressed with prog. counter (PC) is used to fetch individual instructions.

PC is loaded by prog. address (PAGEN). This increments PC.

Pipeline Operation:

Six levels: prefetch, fetch, decode, access, read & execute.

Onchip peripherals:

* S/w programmable wait state generator
  - extend upto 7 m/c cycles to interface with memory & I/O.
* Clk generator - Clk is generated by PLL/internal OSC.
* H/w Timer - 16 bit timing circuit with 4 bit prescalar.
* DMA controller - Transfers data between points in memory
* Host Port Interface (HPI) - Parallel port. Provides an interface to a host processor. Information is exchanged between C54x & host processor.
* Serial ports: 1) Sync. 2) buffered 3) multichannel buffered. 4) Time division multiplexed.

Addressing Modes:

1) Immediate - handle constant data. LD #80H, A
2) Indirect - Uses aux. registers to hold address of operands in memory. To select aux. register, aux. register pointer (ARP) is loaded with 0 to 7.

Types: 1) Auto increment 2) Auto decrement
3) Post indexing by adding contents of AR0.
4) Post indexing by subtracting contents of AR0.
5) Single indirect addressing with no increment
6) Single indirect addressing with no decrement
7) Bit reversed addressing.

3) Register Addressing - Uses operands in CPU reg.
1) Block Move Address Register (BMAR)
2) Dynamic Bit Manipulation Register (DBMR).
ex: BLDP, BLPD.

4) Memory mapped Register - To access CPU & on chip peripheral registers.
LAMM - Load Acc. with Memory mapped register
LMMR - Load Memory Mapped Register
SAMM - Store Acc. in memory mapped register
SMRR - Store memory mapped register.

5) Direct Addressing - Allows CPU to access operands by specifying offset from base address.

6) Circular Addressing - Convolution, Correlation, FIR filtering use circular buffers in memory.
CBSR1 - Circular Buffer 1 Start register
CBSR2 - " 2 "
CBER1 - " 1 End "
CBER2 - " 2 "
CBCR - " " Control register.

Instruction Set:

Arithmetic Instructions:

ADCB - Add Acc. B & carry bit to Acc.
ABS - Absolute value of Acc
ADD - Add data memory value.
ADDB - Add Acc B to Acc.
ADDC - Add data memory & carry bit to Acc.
AND - And data with ACCL, zero ACCH.
BSAR - Barrel shift Acc. right
CMPL - 1's complement Acc.
LACB - Load Acc to Acc. B
LACC - Load data memory with left shift to Acc.
NEG - 2's complement Acc.
NORM - Normalize Acc.
OR - OR memory value with ACCL
ROL - Rotate Acc left by 1 bit.
ROR - Rotate Acc right by 1 bit.
SACB - Store Acc. in AccB
SAMM - Store ACCL in memory mapped register
SBB - Subtract Acc B ~~from~~ Acc.
SFL - Shift Acc. left 1 bit
SFR - Shift Acc. right 1 bit.
SFLB - Shift Acc B & Acc left 1 bit
SFRB - Shift Acc B & Acc right 1 bit.
SUB - Subtract data with left shift from Acc.
XOR - XOR data memory with ACCL.
XORB - XOR data ACCB with Acc.
ZALR - Zero ACCL & load Acc H with rounding

Parallel Logic Unit Instructions:

APL - AND data memory with DBMR.
CPL - Compare data memory with DBMR.
OPL - OR data memory with DBMR.
SPLK - Store long immediate in data memory location
XPL - XOR data memory with DBMR.
LPH - Load data memory with PREG high byte.
LT - Load data memory to TREGO.

TREGO, PREG & multiply instructions:

LTA - Load data memory to TREGO; add PREG with shift to ACC.
LTD - Load data memory to TREGO; Store PREG with ACC & move data.
LTP - Load data memory to TREGO; Store PREG in ACC
MAC - Add PREG with shift to ACC.
MPY - Multiply data memory value by TREGO & Store it in PREG
MPYA - Add PREG to ACC
MPYS - Subtract PREG from ACC.
MPYU - Multiply Unsigned PREG to ACC.
PAC - Load PREG to ACC.
SPAC - Subtract PREG from ACC.
ZPR - Zero PREG.

Branch & Call Instructions:

B - Branch unconditional to prog. memory location.
BACC - Branch conditional to prog. memory location by ACCL
BACCD - Delay conditional to prog. memory loca by ACCL
BANZ - Branch conditional if AR not zero.
BANZD - Delay branch conditional if AR not zero

BCND - Branch conditional to prog. memory location.
BCNDD - Delayed branch conditional to prog. mem. locatn.
BD - Delayed branch unconditional.
CALL - Call to subroutine unconditional.
CALA - Call to subroutine by ACCL.
Conditional: INTR, NMI, RET, RETC.
RETI - Return from interrupt.
XC - Execute next instruction conditionally.

Control functions:

BIT - Test bit
SETC - Set Carry bit
CLRC - Clear Carry bit
IDLE - Idle until nonmaskable interrupt.
NOP - No Operation
RPT - Repeat.

Features of DSP Processors:

- Multiple registers.
- Multiple operands fetching capacity.
- Circular buffers
- Multiple pointers to support multiple operands.
- Multi processing ability.
- Powerful interrupt structure & Timers.